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ELECTRONIC ANALOG COMPUTERS

ELECTRONIC ANALOG COMPUTERS

(D-c Analog Computers)

by

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To the memory of Dean Roland George Dwight Richardson, who has contributed so much to the development of applied mathematics in the United States, this book is respectfully dedicated.

PREFACE

D-c analog computers are relatively simple electronic devices now commonly in use to solve a variety of problems in applied mathematics. The application of such computers to the design of modern automatic control systems has been particularly successful.

The authors hope that the present work will satisfy the growing need for a book on this new subject. The techniques described in this book have become well established during the last five years. Although interest in electronic computation is growing, many engineers lack access to reliable technical information in this field. Moreover, many who do know the details of computer design lack experience in intelligently applying the new techniques to their problems. There are, in particular, the questions of how to set up problems on a computing machine and how to administer a computer installation.

No mathematics beyond that taught in an elementary course on differential equations is required to read this book. The large amount of material which had to be covered has made it impossible to add a chapter on elementary electronics; the reader is, instead, referred to good elementary texts on this subject.

Chapter 1 is an introduction to d-c analog computers. The reader will find the following technical chapters more useful if he is first given an opportunity to see the picture as a whole. Accordingly, Chapter 1 briefly treats the applications of computing devices, the mathematical relations involved, and the techniques used, with the object of gaining an over-all view of the subject.

Chapter 2 attempts to teach a simple, standard procedure for setting up a d-c analog computer to solve a given set of differential equations. The procedure described has proved useful in several years of practical work.

The principles underlying the operating of analog computers are essentially simple. On the other hand, the authors have worked with computer operators in various parts of the country, usually men with degrees in engineering and mathematics, and they have seen many others at work. Many of these people had fine backgrounds ranging from electronics to relativity theory. Thus, although they were new in the computer field, they learned the principles very quickly. *But their chief difficulty, their most formidable source of errors, was in the determination of scale factors, particularly where changes of the time scale are involved.* For some reason, even well-qualified engineers and scientists seem prone to commit scale-factor errors! Every effort has been made to present this subject in detail. All sections in this chapter have been documented with practical examples.

Chapter 3 presents computer setups for a number of practical problems which serve as illustrations of modern industrial d-c analog-computer applications. The preparation of useful records of each computation is particularly stressed. It is hoped that the samples of problem-preparation records shown will be useful for reference.

Chapters 4 to 6 describe the design of electronic computing elements for maximum accuracy, according to the best modern engineering practice. Some of the more technical parts of the accuracy analysis have been placed in the Appendix for the convenience of the reader.

Specifically, Chapter 4 describes the design of the networks and feedback circuits used to perform the mathematical operations of addition, multiplication by constant coefficients, differentiation, and integration, with special emphasis on the design of accurate d-c integrating devices.

Chapter 5 deals with the design and operation of modern d-c amplifier circuits for computer applications. A discussion of the stabilization of high-gain feedback amplifiers is included. Automatic balancing circuits are described, and the actual circuit diagrams of many d-c amplifiers used in modern d-c analog computers are presented.

Chapter 6 discusses developments in the still open fields of analog multiplication and function generation. Several previously unpublished schemes for multiplication and function generation are discussed.

Chapter 7 describes the design of the auxiliary components, such as power supplies and recorders; two complete sections are devoted to a discussion of improved control circuits for d-c analog computers. The remaining portion of this chapter deals with procedures for operating and checking d-c analog computers.

Chapter 8 describes the design and construction of complete computer installations for various applications in industry and science. The choice of computing elements and associated equipment is discussed, with emphasis on reliable and economical operation. Detailed descriptions of several modern d-c analog-computer installations are given, and these installations are discussed in the light of the design principles previously presented.

It was originally intended to include a chapter dealing with questions of computer utilization, personnel, and administration and to show how a computer laboratory may be fitted smoothly into an industrial or scientific organization; this has been prevented by a lack of space.

The present volume is intended primarily as a textbook on d-c analog computers. An effort has been made to keep the text relatively simple technically. In order to bring out certain important facts about analog computers, these facts have been stated repeatedly in different chapters.

The authors regret that the continuation of wartime security regulations makes it impractical to recount the history of analog-computer development and thus to give credit to the many distinguished groups and individuals, both in this country and in England, who have contributed in this field.

The authors are happy to be able to include in this book some of the experiences of many workers and laboratories concerned with the design and application of analog computers and hope that the many circuit diagrams of actual equipment will be found useful. It is believed that this information on tested and tried circuits will enable any reasonably qualified electronic engineer to design d-c analog computing equipment. The authors wish to express their gratitude to the following organizations and individuals who have permitted them to quote from engineering reports and to publish descriptions and photographs of their equipment:

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[1] Now Supervisor of Instrumentation, Cook Research Laboratories, Chicago, Ill.

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CHAPTER 1

INTRODUCTION TO D-C ANALOG COMPUTERS

1.1. The Need for Automatic Computing Devices. Digital and Analog Computers. *Introduction.* The solution of a large number of scientific and engineering problems is possible only after detailed numerical computations. Such computations frequently present formidable practical difficulties. Sometimes, the mathematical relations in question are intricate and involved; again, the sheer quantity of material to be handled may seem to constitute an insurmountable obstacle. Thus, all too often a lack of time, funds, or skilled personnel has actually prevented the useful conclusion of promising scientific and engineering developments.

The advent of vast and very complicated war-born research and development projects immeasurably stressed the importance of obtaining scientific and design information from complex numerical computations. It was, moreover, imperative to obtain such information at high speed and in convenient form in order to satisfy increasing needs for numerical results. The latest and most elegant methods of electronics and automatic control engineering, often developed in the course of the same war projects, were called into play for the design of powerful, high-speed automatic computing machinery.

Digital Computers. Two separate fundamental approaches to this problem are possible, depending on the applications and the accuracy required. *Digital computers* are familiar in the form of desk calculators and such electronic machines as the ENIAC and the EDVAC.[1] Essentially, they consist of counters registering and adding in discrete steps. In the larger calculators programming systems transmit counting impulses between counters in a manner prescribed by the nature of the problem and control the sequence of the computation. Most digital computers perform mathematical operations by combinations of additions. For example, multiplication is performed by repetitive additions, and integration is by summation; converging series may be used in place of trigonometric functions.

The necessity of reducing mathematical operations to arithmetical ones requires considerable planning before a digital computer can handle a problem. Also, because the indirect computations must usually be carried to more places than are required in the final result, the computing elements must have the capacity for dealing with large numbers. Consequently, electronic digital computers find their greatest utility in large-scale computation projects. The fullest utilization of digital computers is obtained in working problems that require repetitive operations of high accuracy.

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[1] Korn, G. A., Elements of D-c Analog Computers, *Electronics*, April, 1948.

Analog Computers. Slide rules, a-c calculating boards, and differential analyzers are *analog computers.* In these devices physical quantities, such as voltages and shaft rotations, are made to obey mathematical relations comparable to those of the original problem. The physical quantities, or *machine variables,* which may be conveniently varied and measured in the laboratory, must, then, behave in a manner analogous to that of the original variables. The magnitudes and the time scale of the analogous variables can be reduced or increased in order to facilitate their measurement.

Analog computers may be grouped into types (mechanical, electrical, electromechanical, etc.) according to the nature of the physical quantities used to represent the mathematical variables. The adaptability of a given type of analog computer to a given problem depends on the availability of *computing elements* to establish the desired mathematical relations between the machine variables chosen.

Accuracy. In general, analog computers are less accurate than digital computers, but the analog machines are usually much cheaper and are often easier to construct. Analog computers are, thus, very useful in many engineering applications requiring accuracies between 0.5 and 5 per cent. The costs of analog computers with accuracies better than 0.1 per cent rise sharply for even small improvements in accuracy. For computations requiring very high accuracy, the use of digital computers is thus indicated. Figure 1.1 shows a qualitative comparison of the cost of analog and digital computers for various accuracies and sizes of installation.

D-c Analog Computers. In d-c analog computers, the variables are represented by d-c voltages which may vary with time. Time is generally used as the independent variable. These computers are capable of solving complicated mathematical problems by means of such simple electrical circuits as d-c amplifiers and potentiometers.

Compared with most mechanical and electromechanical analog computers, the d-c analog computers are relatively cheap, easy to operate and maintain, and capable of flexible interconnections by means of simple patch cords. Perhaps the outstanding advantage of a d-c analog computer is the fact that it can usually be constructed from standard store-bought electronic parts by well-known radio assembly techniques, *without any precision machine shop work.* This fact will be appreciated by any industrial organization which has watched the shop overhead cost rise to 180 per cent of direct labor.

Experience shows that personnel with only routine training in engineering or physics can rapidly learn to use the machines effectively.

Are d-c analog computers effective tools of research and development? If the proper applications are considered, the answer must be an enthusiastically affirmative one.

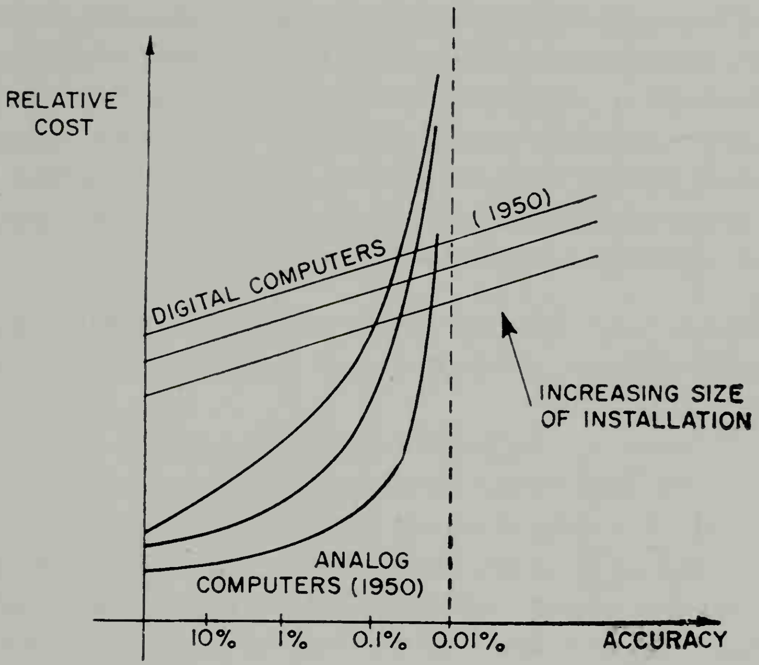

Fig. 1.1. Relative cost vs. accuracy for digital and analog computers (1950) for different size installations.

Practical d-c analog computers range from ultraflexible differential analyzers capable of solving a wide range of different problems[1] to the highly specialized fire-control computers which beat the German V-1 missiles over Antwerp. A special computer for the solution of a given problem may be built on a steel chassis with chrome-plated panel and dials to occupy a place of honor in a laboratory or industrial control center, or it may be wired up from scrap parts on a breadboard at negligible cost and in 2 hours and is equally ready to furnish needed information with fair accuracy. Parameters and coefficients given by the problem, as well as initial values and given functions, can be changed very conveniently by means of calibrated dials on the front panel of the machine. This is particularly useful in many engineering applications where trial solutions for different coefficient settings are used to obtain optimum design parameters.

---

[1] Frost, S., Compact Analog Computer, *Electronics*, July, 1948.

The solution of a set of ordinary differential equations on a d-c analog computer proceeds as follows:

1. With the machine connected up to solve the given problem, the machine variables (voltages) are set to the correct initial conditions prescribed by the problem.
2. The computing elements are then made operative and force the voltages in the machine to vary in a manner prescribed by the given differential equations. The voltage variations with time are recorded and constitute solutions of the given problem.
3. The machine is stopped at a time chosen by the operator. The maximum allowable computing time is usually determined by limitations of the computing elements. The machine is at once reset to its initial condition and is ready for the next run with changed coefficients, initial conditions, etc.

The recorders used for such applications are usually direct-inking recording milliammeters which afford an instantly available, permanent record of the computation.

Repetitive Computers. In an interesting alternative scheme, the steps of the solution outlined above are repeated automatically at a rapid rate (60 cps) through the action of an electromechanical or electronic switching system.[1] During each cycle, each machine variable varies in the prescribed manner and is then reset to its initial value. The results can now be presented on an ordinary cathode-ray oscillograph. The sweep frequency of the oscillograph must be equal to the repetition rate of the computer.

The rapid operation of such "repetitive" differential analyzers constitutes a striking advantage in some applications. In particular it is possible to observe the results of varying coefficient settings and initial conditions at once without having to wait for the completion of a computer "run" as in ordinary computers. On the other hand, the necessity for rapid switching and computation on a fast time scale makes the design of certain computing elements quite difficult. Accordingly, repetitive operation is seldom considered sufficiently reliable for large-scale computations. This ingenious scheme of operation is most useful for quick qualitative estimates of solutions and has been employed very successfully for exploratory runs to avoid unnecessary computations on a larger differential analyzer.

1.2. Representation of Mathematical Relations. The problems solved by analog computers usually deal with the behavior of a number of variables $x_1, x_2, \ldots, x_n$. One such problem may be to find the values of these variables under certain conditions by solving a set of equations. Another problem may be to determine how these variables change with time, or with another independent variable t, under certain conditions. The particular conditions imposed by each problem are usually given as a set of mathematical relations or equations.

---

[1] McCann, G. D., S. W. Herwald, and H. S. Kirschbaum, Electrical Analogy Methods Applied to Servomechanism Problems, *Trans. AIEE*, **65**, February, 1946.

The variables $x_1, x_2, \ldots, x_n$ are represented in the d-c analog computers by corresponding voltages, or *machine variables*, $X_1, X_2, \ldots, X_n$. In general, these machine variables are simply proportional to the corresponding original variables on a convenient scale. The given relations between the original variables are now expressed by an "analogous" set of relations between the machine variables. A straightforward, generally applicable method of obtaining convenient scale factors and the corresponding correct machine equations is discussed in detail in Chap. 2.

Block Representation. Each problem will, then, require certain relations between machine variables to be set up in the computer. Consider first some examples of relations between *two* machine variables:

$$\begin{aligned} X_1 &= X_2 + 15X_2^2 \\ X_1 &= \sin X_2 \end{aligned} \qquad (1.1)$$

In each case, X_1 and X_2 are voltages; and the voltage X_1 is a *function* of the voltage X_2. If X_2 changes, X_1 must also change, in accordance with the given relation.

Figure 1.2 shows schematically how such relations are represented in a d-c analog computer. The voltage X_2 will appear as the *input voltage* of a *block of computing elements* which "takes it into consideration" and produces X_1 as an *output voltage.* This output voltage will be the desired function of the input voltage X_2 if the correct combination of computing elements has been used. Note that, in general, the relation produced by the block in Fig. 1.2 is "unilateral": whereas changes in the input voltage X_2 will produce the correct changes in the output voltage X_1, the converse is not true unless additional connections and computing elements are used.

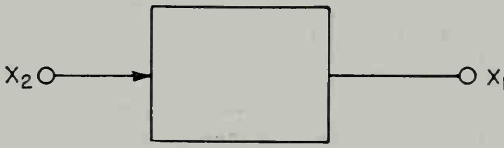

FIG. 1.2. Block representation of functional relationships between machine variables.

Several Variables. More often than not, a voltage X_1 will appear as a function of *several* other machine variables $X_2, X_3, \ldots$:

$$\left.\begin{aligned} X_1 &= X_2 + 19X_3 + \cos X_4 \\ X_1 &= X_2X_3^2 \end{aligned}\right\} \qquad (1.2)$$

The block of computing elements producing X_1 as its output voltage will, then, have more than one input voltage. The effects of the different input voltages, X_2, X_3, . . . are "mixed" to produce the correct functional relation (see Fig. 1.3).

FIG. 1.3. Block representation of functional relationships between machine variables.

Implicit Solution. In relations like

$$X_1 = X_2^2 + 2X_1^3 - X_1^7 \qquad (1.3)$$

it is not possible to solve explicitly for X_1 in order to establish the relation in the computer. But this is not at all necessary. X_1 appears on the right of Eq. (1.3) and may thus be considered as a voltage appearing at an input terminal. This voltage at the input terminal must, of course, always be equal to the voltage X_1 at the output terminal. To ensure this, the output voltage X_1 is treated just as if it were known and is "fed back" to the X_1 input terminal. If suitable computing elements are used, the correct mathematical relation will be established (Fig. 1.4); some care will be necessary in each case to avoid setting up regenerative feedback loops which may result in uncontrolled oscillations. This scheme is useful for *solving certain types of equations* on analog computers (see also Sec. 3.9).

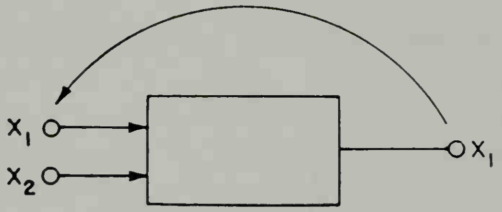

FIG. 1.4. Block representation of functional relationships between machine variables.

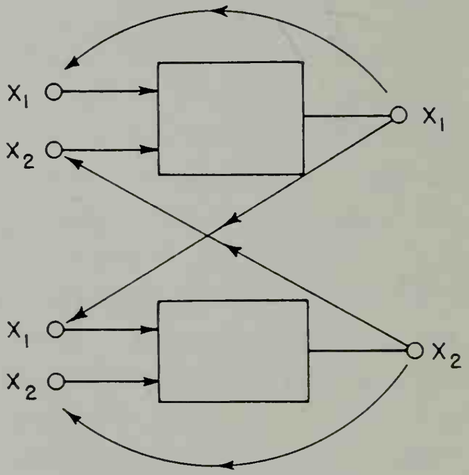

FIG. 1.5. Block representation of functional relationships between machine variables.

Simultaneous Relations. If two or more simultaneous relations between machine variables are given, such as the set

$$\left.\begin{aligned} X_1 &= X_2 + X_1 \\ X_2 &= X_1 + 8X_2^3 \end{aligned}\right\} \qquad (1.4)$$

each equation is "solved" electrically for a different variable. This variable appears, then, as the output voltage of a block of computing elements. Figure 1.5 shows how it is then used as an input voltage for the computing elements establishing the *other* equation(s). The resulting interconnections force the machine variables to satisfy all the given relations.

Time Dependence of Variables. In many problems, it is desired to observe the variations of the machine variables as *functions of the time* τ, which plays the part of the independent variable in the computer. For example, the relations

$$X_1 = X_2 \qquad X_2 = \sin \tau \tag{1.5}$$

describe the variations of two machine variables, X_1 and X_2, with the time τ. Physical time τ here serves as an input to certain computing elements which produce voltages as functions of time. Examples of such devices are potentiometers driven by a clock motor or circuits containing charging and discharging capacitors. The machine variables can thus be made to vary with time in the prescribed manner and may be recorded by means of a recording milliammeter or oscillograph.

Calculus. It is easily seen that a majority of problems in applied mathematics require a description of *changes* of quantities (variables) involved in physical processes. The *differential calculus* is the branch of mathematics concerned with the description of such changes. A *time derivative* like $dX/d\tau$ (derivative of the machine variable X) is a quantitative measure of a change dX: the latter is compared with the corresponding change of a known physical quantity, the *time* τ indicated by a clock. It is practical to write

$$\frac{d}{d\tau} = P \qquad \text{thus} \qquad \frac{dX}{d\tau} = PX \tag{1.6}$$

This notation will be used throughout this book. For example, if $X = \cos \omega\tau$, then

$$PX = P \cos \omega\tau = -\omega \sin \omega\tau$$

Derivatives of machine variables, like PX, are themselves functions of time and may change with time. It is thus usually possible to define the derivative of PX. This is called the *second derivative* of X, $d^2X/d\tau^2 = P^2X$. Higher derivatives are defined accordingly.

Ordinary Differential Equations. Mathematical equations relating measures of change (derivatives) of this type are called *ordinary differential equations.* In many problems a set of such differential equations is given, and it is desired to find the corresponding time dependence of the variables involved. The set of differential equations is *solved* when all the variables are computed as functions of time. Thus the set of differential equations

$$PX = Y \qquad PY = -X \tag{1.7}$$

is satisfied by the functions

$$X = C \sin (\tau + \varphi) \qquad Y = C \cos (\tau + \varphi) \tag{1.8}$$

where C and φ are arbitrary constants.

Figure 1.6 shows how relations of this type may be represented on d-c analog computers. All derivatives of machine variables are represented by voltages in the machine and are, thus, themselves machine variables. Figure 1.6 shows how special computing elements called *integrators* form the time integrals of the voltages proportional to the derivatives PX and PY to obtain the machine variables X and Y, respectively. With this arrangement the machine variables X, Y, PX, and PY must, again, vary as prescribed by the given mathematical relations. Records of X and Y will, then, constitute a solution of the set of differential equations [Eqs. (1.7)]. Analog computers capable of solving sets of ordinary differential equations are often referred to as *differential analyzers*. Solutions of differential equations are often needed and may be very laborious if performed by conventional numerical methods. A differential analyzer can thus be a very useful device indeed.

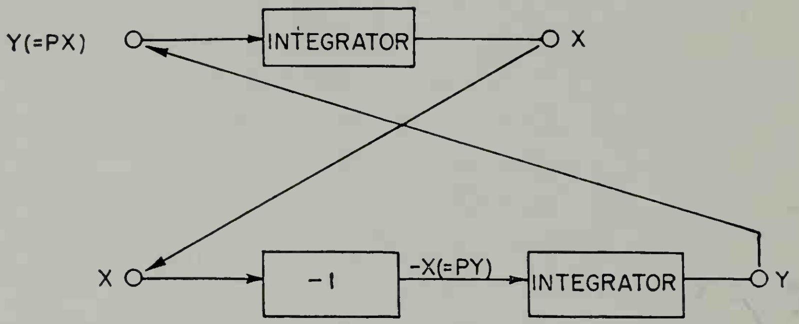

FIG. 1.6. Block representation of a differential-equation solver.

Initial Conditions. In order to obtain definite information from a mathematical relation specifying changes of variables, it is necessary to know the values of the variables at the start of the changes. Because of this simple fact a set of ordinary differential equations cannot be definitely solved unless the *initial conditions* of variables and derivatives corresponding to the start of the computer "run" are known. Specifically, it can be shown[1] that the output voltage of each integrator at the start of the computation must be specified if a definite solution is to be obtained. In the computer setup illustrated by Fig. 1.6, it will be necessary to specify the initial conditions for X and Y. The output voltages of the corresponding integrators must be set to the correct voltages. This will determine the constants C and φ left arbitrary in Eq. (1.8), and a definite

[1] Caldwell, S. H., and V. Bush, A New Type of Differential Analyzer, *J. Franklin Inst.*, October, 1945, p. 255.

solution will be obtained. Every differential analyzer must contain provisions for setting such initial conditions into the computing elements.

An Advantage of Analog Computers. The usefulness of the "block" representation of mathematical relations in practical computer applications cannot be overemphasized. This representation serves as an instructive model which illustrates relationships between different variables of a problem. In many applications, particularly in communications and control engineering, the functional resemblance between the model and the actual equipment described by the mathematical relations is striking. An amplifier or a servomechanism component does have input and output variables related by sets of differential equations. Each such component can, then, be represented by a "block" of computing elements in the machine. The flow of information in the computer will then be closely analogous to the flow of signals in the physical device under study. Blocks of computing elements may be interchanged with others representing different equipment until the optimum combination of components for the physical system is established. It was already mentioned that one or several components of the physical device may take the place of the corresponding block in the computer itself. They may thus be tested, with the computer running on a 1:1 time scale. For such purposes the analog computers are, indeed, better suited than digital computers, which are usually forced to consider an entire problem as a whole for reasons of economy. In the analog computers, the actual relations between variables "become alive" and can be observed directly. The next task is that of designing generally useful building blocks or computing elements.

Computing Elements Needed. It will, in general, be desirable to make a computing machine easy to construct and suitable for a wide variety of applications. Accordingly, it is customary to perform more complicated mathematical operations on the computer voltages (machine variables) through *combinations of a limited number of simple operations, performed by basic computing elements. It has been proved explicitly*[1] *that a very wide range of problems can be solved conveniently by the application of only the following computing elements:*

1. Devices which will multiply a machine variable by a (positive or negative) constant coefficient
2. Devices which will generate the sum of two (or more) machine variables
3. Devices which will generate the product of two machine variables

[1] Murray, Francis J., *The Theory of Mathematical Machines*, King's Crown Press, New York, 1947.

4. Devices which will generate functions of machine variables
5. Devices which will generate either the time integral or the time derivative of a machine variable

Some of the more commonly used devices capable of performing such operations with d-c voltages will be discussed, one by one, in the following section. The design of these and other computing elements will be covered in Chaps. 4 and 5.

1.3. Algebraic Operations. Compared with many other computing devices, d-c analog computers are often strikingly simple. The reason for this is in the relative ease with which the fundamental operations of multiplication by constants, addition, differentiation, and integration can be performed by utilizing simple basic laws of current flow.

Multiplication by a Constant. Figure 1.7*a* shows how a simple voltage divider or potentiometer is used to multiply a voltage by a constant positive coefficient $a(1 \geq a \geq 0)$:

$$X_o = aX_1 \tag{1.9}$$

The coefficient setting may be read on the potentiometer dial. The dial scale will be linear unless the potentiometer is loaded appreciably by another computing element connected to its output terminals (see Sec. 4.1).

D-c amplifiers with feedback make it possible to multiply a d-c voltage by a constant greater than 1. In most practical cases the sign of the output voltage will be the opposite of that of the input voltage. The feedback amplifier thus multiplies by a negative constant; it acts as a *phase inverter.*

D-c amplifiers with various feedback networks are used extensively in d-c analog computers. They are often referred to as *operational amplifiers.* The transfer functions of such devices are determined mainly by the feedback networks; the amplifier forward gain is usually made as high as possible in order to make the transfer function stable. The amplifiers and their associated power supplies are designed for low drift.[1] All input and output voltages are measured with respect to a ground reference. It is clear that the output voltages of operational amplifiers must not exceed certain maximum absolute values (usually plus or minus 100 volts) corresponding to the range of linear amplifier operation.

Consider the arrangement of Fig. 1.7*b*. Let A be the amplifier forward

[1] For an elementary discussion of amplifier terminology, the reader is referred to D. G. Fink, *Engineering Electronics,* McGraw-Hill, New York, 1938. More advanced discussions will be found in Vol. 18 of the MIT Radiation Laboratory Series. Properties desirable in computer amplifiers as well as typical circuits are discussed in Chap. 5. of the present work.

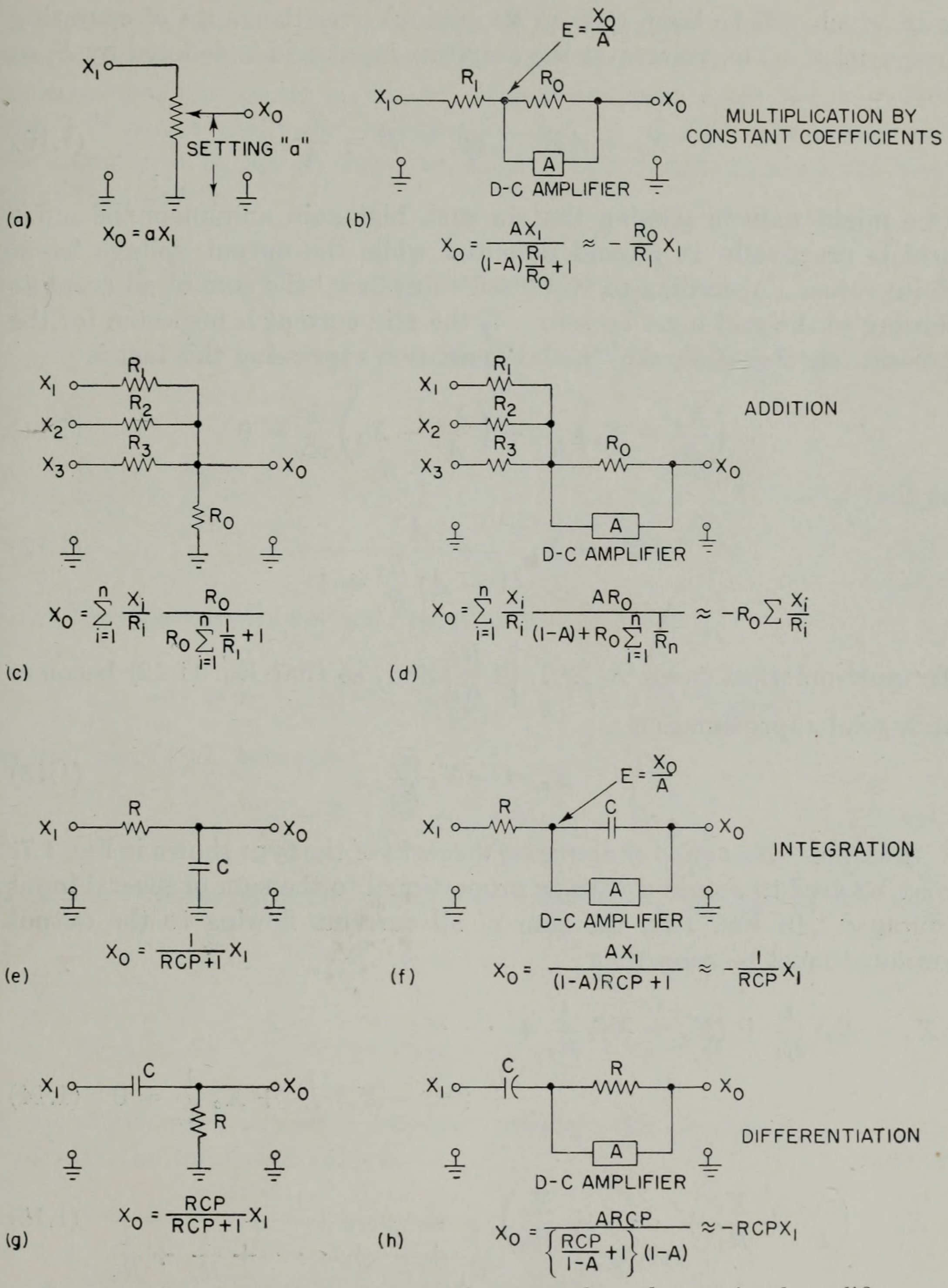
(a)
X_1
X_0
SETTING "a"
$X_0 = aX_1$
(b)
$E = \frac{X_0}{A}$
R_1
R_0
A
D-C AMPLIFIER
$X_0 = \frac{AX_1}{(1-A)\frac{R_1}{R_0}+1} \approx -\frac{R_0}{R_1}X_1$
MULTIPLICATION BY CONSTANT COEFFICIENTS
(c)
X_1
X_2
X_3
R_1
R_2
R_3
R_0
$X_0 = \sum_{i=1}^{n} \frac{X_i}{R_i} \frac{R_0}{R_0 \sum_{i=1}^{n} \frac{1}{R_i}+1}$
(d)
$X_0 = \sum_{i=1}^{n} \frac{X_i}{R_i} \frac{AR_0}{(1-A)+R_0 \sum_{i=1}^{n} \frac{1}{R_n}} \approx -R_0 \sum \frac{X_i}{R_i}$
ADDITION
(e)
R
C
$X_0 = \frac{1}{RCP+1}X_1$
(f)
$X_0 = \frac{AX_1}{(1-A)RCP+1} \approx -\frac{1}{RCP}X_1$
INTEGRATION
(g)
$X_0 = \frac{RCP}{RCP+1}X_1$
(h)
$X_0 = \frac{ARCP}{\left\{\frac{RCP}{1-A}+1\right\}(1-A)} \approx -RCPX_1$
DIFFERENTIATION

Fig. 1.7. Summary of simple computing networks and operational amplifiers.

gain which will be large (500 to 50 million) over the range of operating frequencies. The voltage at the amplifier input grid is denoted by E, so that

$$X_o = AE \qquad \text{or} \qquad E = \frac{X_o}{A} \tag{1.10}$$

One might note in passing that in such high-gain amplifiers the input grid is practically at ground potential while the output voltage has a finite value. According to Kirchhoff's first law,[1] the sum of all currents flowing to the grid must be zero. If the grid current is neglected for the present (see Sec. 4.3), the "nodal" equation expressing this fact is

$$\left(\frac{X_o}{A} - X_o\right)\frac{1}{R_o} + \left(\frac{X_o}{A} - X_1\right)\frac{1}{R_1} = 0 \tag{1.11}$$

so that

$$X_o = X_1 \frac{A}{(1 - A)\dfrac{R_1}{R_o} + 1} \tag{1.12}$$

In most practical cases $|A| \gg 1$, $\left|A\dfrac{R_1}{R_o}\right| \gg 1$, so that Eq. (1.12) becomes to a good approximation

$$X_o = -X_1 \frac{R_o}{R_1} \tag{1.13}$$

Addition. Summing or averaging networks of the type shown in Fig. 1.7*c* may be used to obtain a voltage proportional to the sum of several input voltages. In Fig. 1.7*c*, the sum of all currents flowing to the output terminal must be zero, thus

$$(X_o - X_1)\frac{1}{R_1} + (X_o - X_2)\frac{1}{R_2} + \cdots + (X_o - X_n)\frac{1}{R_n} + X_o\frac{1}{R_o} = 0 \tag{1.14}$$

so that

$$X_o = \left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right)\frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \cdots + \dfrac{1}{R_n} + \dfrac{1}{R_o}} \tag{1.15}$$

Such networks may be used for addition, but it is seen from Eq. (1.15) that the performance changes appreciably with the load resistance R_1 and that there is considerable attenuation in this circuit.

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[1] Smythe, W. R., *Static and Dynamic Electricity*, McGraw-Hill, New York, 1939.

Figure 1.7*d* shows a feedback amplifier arrangement which permits the addition of several voltages or machine variables as well as multiplication by constants and phase inversion. The operational amplifier shown in Fig. 1.7*d* has several input voltages $X_1, X_2, \ldots, X_n$. The voltage at the input grid is, again, equal to X_o/A. The nodal equation for this point becomes

$$\left(\frac{X_o}{A} - X_1\right)\frac{1}{R_1} + \left(\frac{X_o}{A} - X_2\right)\frac{1}{R_2} + \cdots + \left(\frac{X_o}{A} - X_n\right)\frac{1}{R_n} + \left(\frac{X_o}{A} - X_o\right)\frac{1}{R_o} = 0 \quad (1.16)$$

so that

$$X_o = \left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right)\frac{AR_o}{(1 - A) + R_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}\right)} \quad (1.17)$$

With high-gain amplifiers one has usually

$$|A| \gg 1, \qquad |A| \gg R_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}\right)$$

so that Eq. (1.17) becomes

$$X_o = -R_o\left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right) \quad (1.18)$$

Equation (1.18) is particularly useful for establishing relations of the form

$$X_o = -(a_1X_1 + a_2X_2 + \cdots + a_nX_n) \quad (1.19)$$

with

$$a_1 = \frac{R_o}{R_1} \qquad a_2 = \frac{R_o}{R_2} \qquad \cdots \qquad a_n = \frac{R_o}{R_n}$$

A commonly used *summing amplifier* employs the arrangement of Fig. 1.7*d* with the resistance values

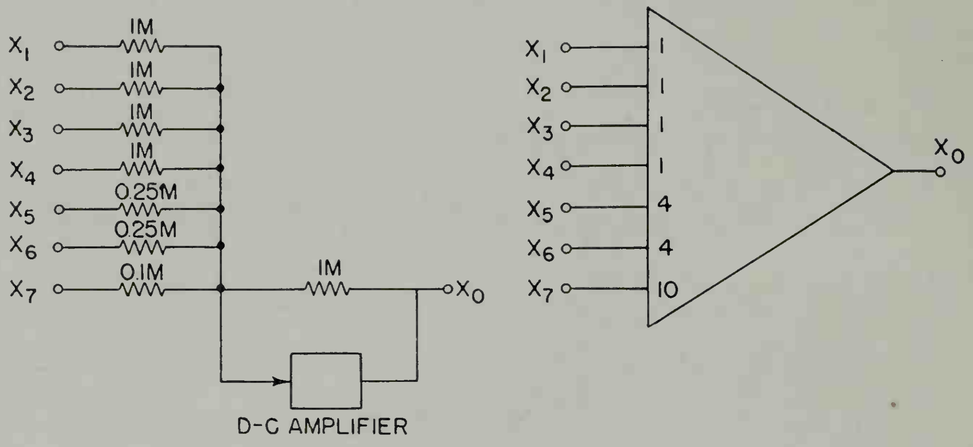
$R_o = 1$ megohm
$R_1 = R_2 = R_3 = R_4 = 1$ megohm
$R_5 = R_6 = 0.25$ megohm
$R_7 = 0.1$ megohm

so that (1.19) becomes

$$X_o = -(X_1 + X_2 + X_3 + X_4 + 4X_5 + 4X_6 + 10X_7)$$

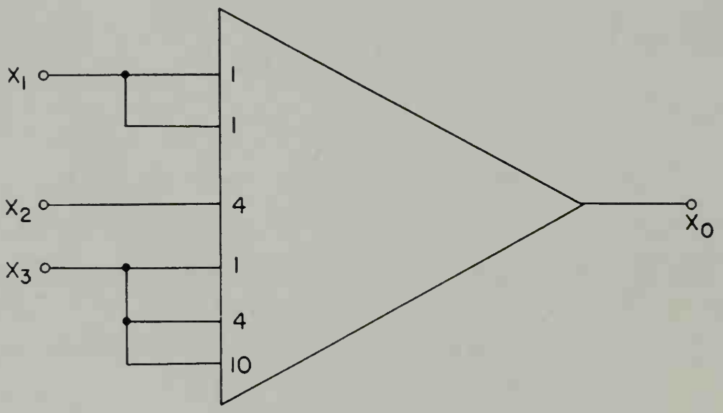
Seven input terminals corresponding to the coefficients 1, 1, 1, 1, 4, 4, and 10 are available. Figure 1.8 shows this arrangement and the commonly used block-diagram symbol for such a summing amplifier.

A summing amplifier may be used for multiplying a variable by a constant coefficient in yet another manner. It is possible to connect the same input voltage to two or more input terminals of the summing amplifier; the variable is thus added to itself (or several products of the variable by constant coefficients are added). Figure 1.9 illustrates this use of "parallel inputs" with a summing amplifier. Summing amplifiers and potentiometers can be combined to form more complicated sums and differences.



$$X_0 = -(X_1 + X_2 + X_3 + X_4 + 4X_5 + 4X_6 + 10X_7)$$

FIG. 1.8. Summing amplifier and block-diagram symbol.



$$X_0 = -(2X_1 + 4X_2 + 15X_3)$$

FIG. 1.9. Summing amplifier, illustrating use of parallel inputs for multiplication by constant coefficients.

Multiplication. The design of computing elements whose output voltage is the *product* of two input voltages is somewhat more difficult.

Chapter 6 lists a number of devices capable of performing this function by electromechanical or electronic means. It will suffice here to mention the most commonly used of these multipliers. Consider an ordinary linear potentiometer of the type described above with an input voltage X_1. If the potentiometer setting a is made proportional to a second input voltage X_2, then the output voltage aX_1 must be proportional to the product of X_1 and X_2. A *servomechanism* consisting of an amplifier or relay and an electric motor can be made to position the potentiometer shaft in the desired manner. The voltage X_2 is applied to the input terminal of the servomechanism, and the servomotor will turn the potentiometer shaft continuously to the correct setting (Fig. 1.10).

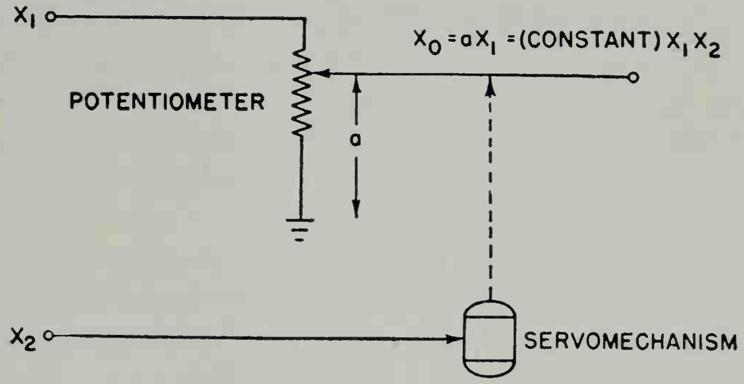


FIG. 1.10. Servo multiplier.

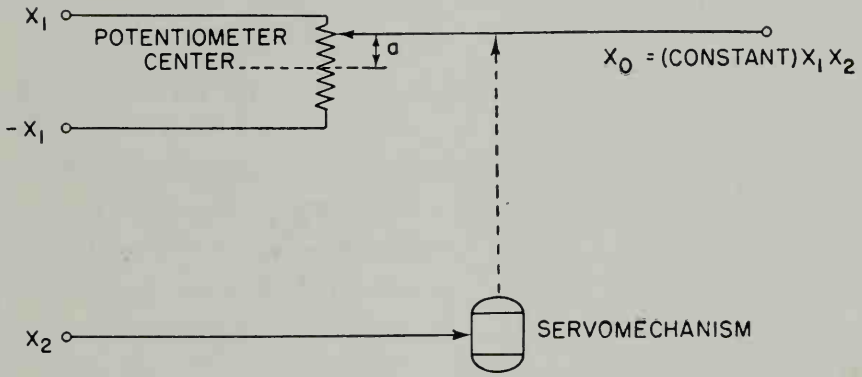


FIG. 1.11. Servo multiplier.

This arrangement permits multiplication by positive voltages X_2 only. In order to permit X_2 to assume negative as well as positive values, it is customary to apply the voltages X_1 and $-X_1$ to two ends of the potentiometer. The output voltage X_o is then still aX_1, and a varies from -1 through 0 to $+1$ as the potentiometer shaft is turned. It is thus possible for a to be proportional to negative as well as positive values of X_2, and the output voltage will be proportional to the product X_1X_2 for all combinations of signs (Fig. 1.11).

Function Generation. With properly tapered and tapped potentiometers, the output voltage can be made equal to a nonlinear function of the shaft setting a and thus of the second input voltage X_2:

$$X_o = X_1 a = X_1 a(X_2) \qquad (-1 \leq a \leq 1) \tag{1.20}$$

Such servo-driven *function generators* permit the multiplication of a voltage (constant or variable) by a function of another voltage. For example, it is possible to *divide* a voltage X_1 by another voltage X_2 by performing a multiplication by the function $1/X_2$. A number of other circuits for division are discussed in Sec. 6.3. With all servo-driven devices, the speed of the computation is limited by the speed with which the motor can follow the variations of the voltage X_2.

1.4. Electronic Integration and Differentiation. Integrations and differentiations of d-c voltages are usually performed with respect to the time τ as the independent variable.

The integrators described in this book permit integration with respect to the time τ. If integration of a machine variable X with respect to another machine variable, Y, say, is desired, use must be made of the relation

$$\int X\,dY = \int X \frac{dY}{d\tau}\,d\tau = \int XPY\,d\tau \tag{1.21}$$

which can be implemented through the use of an integrator and a multiplier.

The physical basis of most electrical integrating and differentiating devices is a simple property of capacitors. The current through such a capacitor (Fig. 1.12) is always proportional to the derivative of the voltage E across the capacitor:

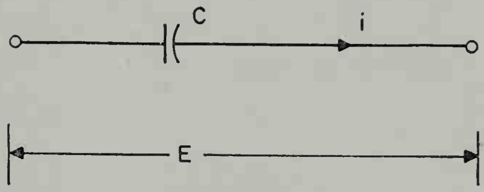


FIG. 1.12. Illustrating the basic property of capacitors.

$$i = CPE \tag{1.22}$$

In the simple *integrating network* shown in Fig. 1.7*e*, an input voltage X_1 will start to charge the capacitor C through the resistor R in a manner amounting approximately to integration. Unfortunately the voltage built up on C during this process can oppose the input voltage and prevent accurate computation. Quantitatively speaking, the sum of all currents flowing to the output terminal must be zero, so that

$$(X_o - X_1)\frac{1}{R} - X_o CP = 0$$

and

$$X_o = \frac{1}{RCP + 1} X_1 \tag{1.23}$$

The *transfer function* X_o/X_1 of a true integrator would be

$$\frac{X_o}{X_1} = \frac{const}{RCP} \tag{1.24}$$

Figure 1.13 shows the response of an integrating network to an input voltage step (a constant-voltage source suddenly turned on). The integrating network will, in all cases, approximate the performance of a true integrator reasonably well for a short time after the computation is started. The transfer function of (1.23) approximates that of a true integrator more closely if the values of C and R are increased, but this also increases the attenuation.

Electronic Integration. The *electronic integrator* shown in Fig. 1.7*f* is more accurate and practical than the simple integrating network. Good integration is obtained with little or no attenuation and smaller values of resistance and capacitance. Moreover, the output can be made to match a low-impedance load, such as a potentiometer, without impairing the quality of the integration.

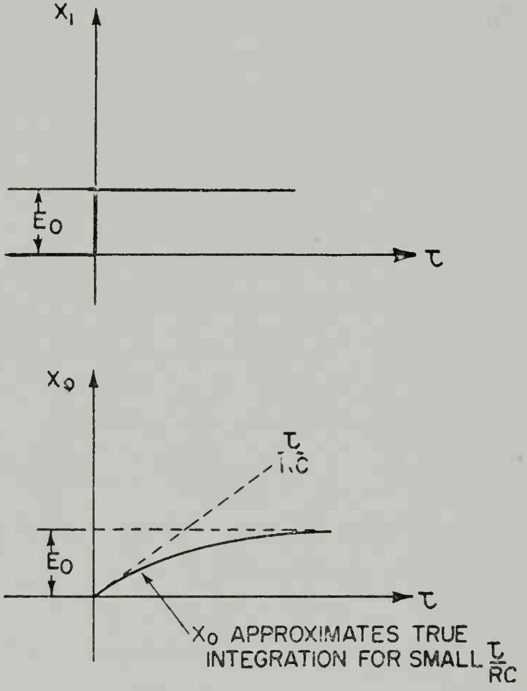


FIG. 1.13. Integration of step-function voltage by a simple integrating network.

With high-gain feedback amplifiers the input grid voltage $E = X_o/A$ will be quite small for high values of the amplifier gain A. Since now only this small voltage can oppose the input voltage X_1, the amplifier will charge the capacitor at a rate quite closely proportional to X_1. The output voltage X_o will then be closely proportional to the time integral of input voltage X_1. The transfer function of the electronic integrator is derived precisely in the same manner as that of other operational amplifiers. The nodal equation for the input grid is

$$\left(\frac{X_o}{A} - X_1\right)\frac{1}{R} + \left(\frac{X_o}{A} - X_o\right)CP = 0 \tag{1.25}$$

if capacitor leakage is neglected for the present. Thus

$$X_o = \frac{A}{(1 - A)RCP + 1} X_1 \tag{1.26}$$

For large values of A and reasonably short computing times, this is approximated by

$$X_o = -\frac{1}{RCP} X_1 \tag{1.27}$$

which corresponds to integration.

Since Eq. (1.27) is only approximately true, there is still a limitation on the allowable computing time. A comparison of (1.26) and (1.23) shows, however, that a considerable improvement has been effected. Because of the nature of integration, it is possible for initial errors in integration to increase with time. Accordingly the design of the integrators is a crucial part of the entire computer design. The detailed design procedure as well as estimates of maximum allowable computing times will be discussed in Sec. 4.3.

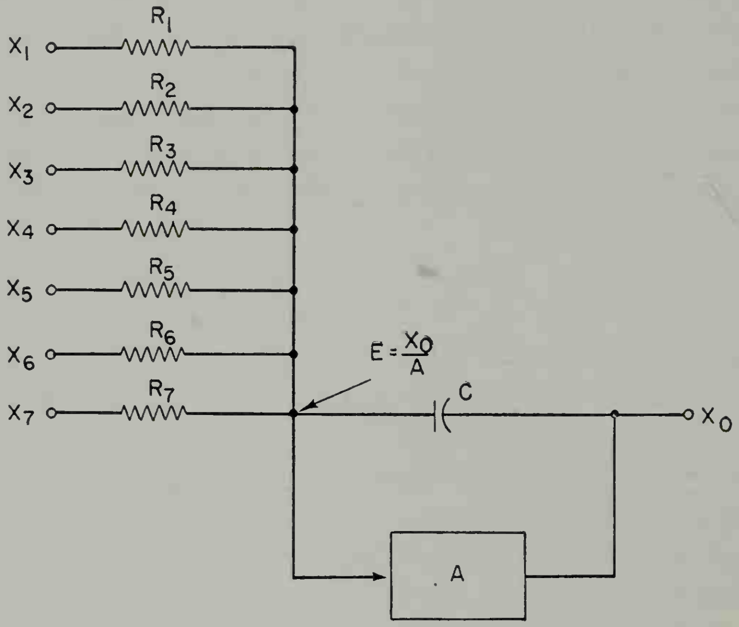


Fig. 1.14. Summing integrator.

Figure 1.14 shows an electronic *summing integrator*. This operational amplifier is capable of summing as well as integration. The output voltage of the summing integrator is, to a similar approximation as (1.27),

$$X_o = -\frac{1}{CP}\left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right) \tag{1.28}$$

Electronic Differentiation. Networks and operational amplifiers capable of approximate differentiation with respect to time may be constructed by reversing the positions of resistors and capacitors in the corresponding integrators, as illustrated in Fig. 1.7*g* and *h*, which also show the respective transfer functions. Computer designers have shied away from the use of differentiators because

1. Differentiators have a tendency to amplify noise pulses.
2. Their inherently good high-frequency response may lead to unwanted high-frequency oscillations in the computing machine unless special precautions are taken.

It is generally possible to use integrators instead of differentiators in differential equation solvers. In all such machines, whether integrators or differentiators are used, some provision must be made for setting the given initial conditions into the machine. Several methods for accomplishing this are described in Sec. 7.1.

1.5. Complete D-c Analog-computer Installations. Applications. By judiciously combining basic computing elements of the types discussed in the last section, it is possible to design complete d-c analog computers for a wide range of applications. The technical details of the economical design of such installations will be discussed in Chap. 8.

A complete d-c analog computer will, in general, comprise the following components in one form or another:

1. *Linear electronic computing elements* (operational amplifiers and associated networks). These will perform the operations of amplification, phase inversion, differentiation, and integration, as necessary. The cost and complication of these and other computing elements will depend on the required accuracy.
2. *Potentiometers or attenuators* to set constant coefficients.
3. *"Nonlinear" electronic or electromechanical computing elements* for performing operations such as multiplication, division, function generation, and limiting. Such devices are needed only if the problems to be solved call for nonlinear operations.
4. Permanent wiring or flexible means for *interconnecting* these computing elements as needed for the problems in question. Telephone patch cords and jacks are a practical and convenient means for such interconnections.
5. *Control circuits* for starting the computation with the correct initial condition settings and for stopping the machine after the computation is completed. Such circuits must be designed for reliable as well as convenient operation.

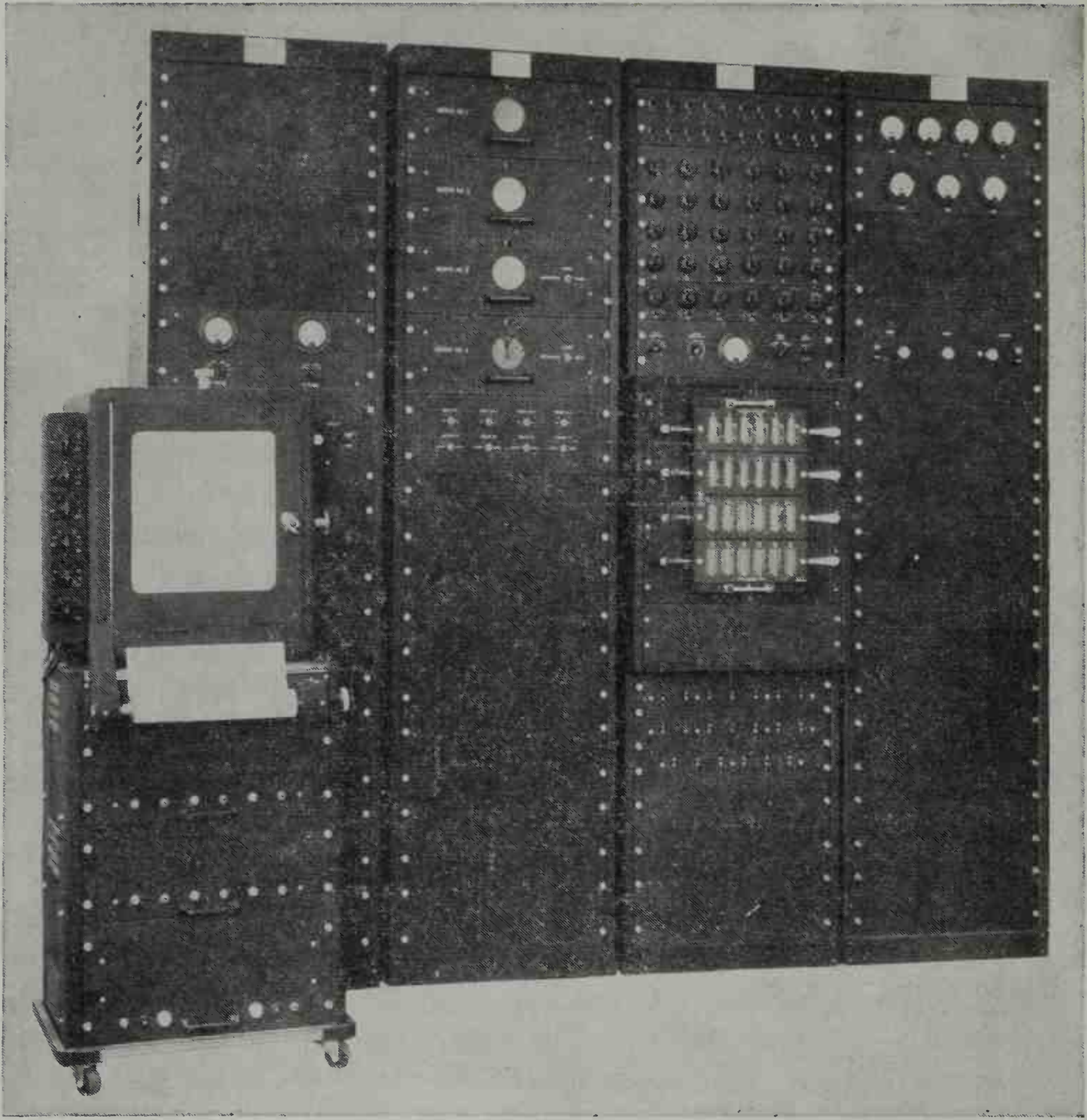


FIG. 1.15. Late model REAC computer (Reeves Instrument Corporation, New York City). The cabinets, from left to right, contain the recorder and driver amplifiers, servo power supplies, computer servomechanisms, potentiometers and operational amplifiers, and power supplies for the d-c amplifiers. Note the patch boards used for multishift operation of the central patch bay (second cabinet from the right; see also Secs. 8.3 and 8.6).

6. Means for *recording or measuring* the d-c voltages in the machine. Recording milliammeters are commonly used to furnish a permanent inked record of voltages against time.
7. Regulated *power supplies* which furnish the voltages needed for the operation of the electronic components.

In addition to special-purpose computers assembled permanently or on a breadboard for the solution of single problems, an institutional or industrial laboratory may utilize a flexible electronic differential analyzer of small to moderate size (10 to 20 operational amplifiers) for the solution of general mathematical problems which present themselves in the course of research or development work. Such computers can be maintained and serviced by any alert radio technician without special preparation. Experience has shown that engineers and scientists with an average college background in mathematics can learn the use of an electronic differential analyzer without difficulty and will quickly appreciate the possibilities afforded by this powerful modern tool.

Large-scale analog-computer laboratories, staffed by mathematical experts as well as specialists in the principal fields of application, are now in existence in several parts of the United States. Such laboratories are capable of rendering invaluable services to science and industry by performing computations too extensive in scope or volume to be handled by smaller facilities. Such computer laboratories are usually started by government or industrial grants and may well be self-supporting. An installation of this type may comprise two or more large electronic differential analyzers which can be interconnected and operated together for the solution of especially complicated problems. Test facilities for automatic control machinery to be tested with analog computers may adjoin the differential analyzers, and digital computers may further increase the range of the problems which can be solved.

Figure 1.15 shows a flexible electronic differential analyzer of the d-c analog type. A number of computer laboratories using such machines have been in continuous operation for a number of years and have an enviable record of achievement.

CHAPTER 2

PRACTICAL SETUP PROCEDURE

2.1. Introduction. In order to solve a given set of differential equations on a d-c analog computer it is necessary to make certain voltages in the machine vary with time in a manner prescribed by the given differential equations. The computer is "set up" for the given problem if a suitable arrangement of computing elements ensures the existence of the correct mathematical relations between the voltages in the computer.

It is clearly of the greatest importance for the engineers or mathematicians in charge of a computer laboratory to acquire skill and facility in the technique of setting the machine up for a given problem.

The setup procedure described in the following sections has evolved during several years of practical work. It seems to be sufficiently simple and straightforward to enable even inexperienced personnel to obtain workable setups for a very large number of problems. On the other hand, there are usually many possible setups for a given problem. It will always be somewhat of an art to find the setup which will result in the greatest possible accuracy or in the most economical use of the computing elements. Experience as well as ingenuity will give the operator a feeling for the new mathematical medium, and one may say that an elegant setup will be aesthetically satisfying as well as practically useful.

2.2. Machine Variables and Scale Factors. It is very important to remember that the machine will establish mathematical relations not between the original variables of a problem but between voltages "simulating" these variables. In order to keep this distinction clearly in mind it is useful to denote the voltage or *machine variable* representing a given mathematical variable in the computer by the corresponding capital letter. Thus an angle δ would be represented by a voltage Δ, a velocity v by a voltage V, a variable x by X, etc. Since in many d-c analog computers the machine variables can vary between the limits of plus and minus 100 volts, it has been found convenient to measure the machine variables in units of 100 volts, or *machine units*. These conventions will be adhered to in the following discussions.

The first step in any computer setup must be the establishment of relations describing the transformations from the given problem variables to machine variables and vice versa. In all ordinary cases these *trans-*

formation equations are simply of the form

$$X = a_x x \qquad \Delta = a_\delta \delta \qquad V = a_v v, \text{ etc.} \tag{2.1}$$

The dimensional coefficients a_x, a_δ, a_v, etc., in Eq. (2.1) are called *scale factors*. The scale factor of any variable is numerically equal to the number of 100-volt machine units corresponding to 1 unit of the simulated variable. Each voltage is given, numerically and dimensionally, by the product of scale factor and variable. Conversely, the correct value of any variable can be determined from the corresponding voltage by dividing the latter by the scale factor.

Example. Let δ be an aircraft angle of attack. The transformation equation $\Delta = 4\delta$ indicates that the voltage Δ simulates the angle δ with a scale factor $a_\delta = 4$ (100 volts/radian), so that an angle of ¼ radian or 14.3 deg is represented by 1 machine unit or 100 volts in the machine.

The choice of scale factors is governed by two conflicting considerations between which some compromise must be made: (1) scale factors should be as large as possible in order to minimize percentage errors due to stray voltages, and (2) the machine variables must not exceed plus or minus 100 volts (1 machine unit) so that all tubes operate within the linear portions of their characteristics. Because of these considerations, it is necessary to possess some information as to the expected ranges of the variables in order to limit the scale factors accordingly. Estimates of these ranges may be obtained from a knowledge of the physical background of the problem, from approximate numerical computations, or from the results of trial runs. This information need not be accurate, since it is relatively easy to change the scale factors during the course of the computation.

In general, a scale factor a_x associated with the variable x will be chosen according to the following rule:

$$a_x \leq \frac{1}{\text{maximum expected value of } |x|} \; \frac{\text{machine units}}{\text{unit}} \tag{2.2}$$

This ensures that the largest expected excursion of the machine variable X will never exceed 1 machine unit. In most cases a convenient round number, somewhat less than the numerical value given by Eq. (2.2), may be used for the scale factor. Scale factors smaller than those indicated by the foregoing must be used

1. If it is inevitable that *multiples* of the machine variables in question appear at an amplifier output terminal.
2. If the time scale (see Sec. 2.4) is fixed and the voltages representing *derivatives* of machine variables would overload one or more amplifiers.

Scale factors for different variables will, in general, differ from one another in magnitude and dimensions. As a rule, there will be one transformation equation and thus one scale factor associated with each variable of the problem. Quite frequently, however, a mathematical variable, x, say, may be represented by two or more different voltages or machine variables X_1, X_2, etc., in different parts of the computer. In such cases there may be more than one transformation equation and more than one scale factor for the variable in question.

Functions of Machine Variables. Functions of machine variables such as X^2, $\sin Y$, e^Y, etc., may be considered as new problem variables f_1, f_2, etc. The absolute values of the corresponding machine variables F_1, F_2, etc., should not exceed 1 machine unit.

Examples. If $f_1 = \sin Y$, the corresponding transformation equation can be simply

$$f_1 = F_1$$

If $f_2 = e^Y$ with $-1 \leq Y \leq 1$, the corresponding transformation may be

$$f_2 = eF_2$$

where

$$F_2 = \frac{1}{e} e^Y$$

Note that the new function F_2 can be obtained simply by means of a potentiometer function generator whereas the original function f_2 cannot (see also Secs. 2.8 and 6.5)

It is not really necessary to write the above relation explicitly in each case; it will usually suffice to rewrite the machine equation in the manner shown in the following example.

Example. The machine equation

$$PX = \frac{1.5}{X} \qquad \text{with } 0.1 < X < 1$$

can be rewritten in the form

$$PX = 15\left(\frac{1}{10X}\right)$$

The function in parentheses can be generated by means of a function potentiometer.

Recorder Scales. The adjustment and choice of the *recorder scales* before computation may be properly considered as part of the computer operation as such and are described in Sec. 7.3.

2.3. Choice of Time Scale. If the transfer functions of all computing elements are given in terms of real time units, a d-c analog computer may be considered as capable of integrating and differentiating the machine variables with respect to real time *and only with respect to real time.* In other words, *real time, henceforth denoted also as machine time*

τ, *is the independent machine variable.*[1] The *time scale* is established by writing a transformation equation relating the machine time τ to the mathematical independent variable, t, say,

$$\tau = \alpha_t t \tag{2.3}$$

From (2.3) it follows that

$$p = \frac{d}{dt} = \alpha_t \frac{d}{d\tau} = \alpha_t P \tag{2.4}$$

where the operator $P = d/d\tau$ denotes differentiation with respect to the real or machine time τ. The dimensional coefficient α_t is called the *time-scale factor* and is often referred to simply as the time scale. The time-scale factor is numerically equal to the number of seconds representing the unit of the independent variable under consideration. Again, the value of the independent variable at any instant of the computation is obtained by dividing the elapsed computing time by the time-scale factor.

Example. Let the time t be the independent variable in a physics problem. *Do not confuse τ and t!* The transformation equation $\tau = 10t$ indicates that the machine time "simulates" the problem time t with a time-scale factor of 10. Thus a problem time of $\frac{1}{10}$ sec is represented by a machine time of 1 sec. All rates of change in the problem have been *slowed down* in the ratio α_t:1 or 10:1.

The nature of a d-c analog computer makes it easy to change the time scale of a computer run within certain limits. When the time scale is changed, the voltage changes in the computer remain proportional to the corresponding changes of the mathematical variables, but the rates at which these voltage changes take place may be speeded up or slowed down in order to improve the accuracy or increase the convenience of the computation.

The *choice of the time scale* is, again, governed by a number of conflicting considerations, which may be summarized as follows:

1. Almost all errors associated with d-c integrators tend to increase with time (see Sec. 4.3) and are thus minimized by short computing times (small α_t).
2. Computer servos, on the other hand, may become unreliable at frequencies higher than 2 cps (see Sec. 6.4). From this point of view, a slow time scale seems to be more favorable if servos are to be used.

---

[1] While the symbol t for real time might be preferable, its use is not practical because of the frequent use of t as the independent variable in physics problems. It is important to consider the meanings of the two symbols τ and t very carefully so that the two may not be confused.

3. With a small time-scale factor all rates of changes and therefore the coefficients in the machine equations are increased. This may make it necessary to cascade amplifiers to the point of instability and also to reduce the scale factors of certain variables whose derivatives would otherwise overload the computer.

Example. Consider the differential equation

$$\frac{dy}{dt} = py = 50$$

Suppose the scale factor for y has been chosen so that

$$a_y = \frac{1}{200}\left(\frac{\text{machine units}}{\text{unit}}\right)$$

or

$$y = 200Y$$

Then for $\tau = t$ (1:1 time scale), the machine equation is

$$Y = \frac{1}{P}\left(\frac{1}{4}\right)$$

For $\tau = 100t$ ("slow" time scale = 100:1)

$$Y = \frac{1}{P}\left(\frac{1}{400}\right)$$

But for $\tau = \frac{1}{100}t$ ("fast" time scale = $\frac{1}{100}$:1)

$$Y = \frac{1}{P}(25)$$

and the voltage of −25 machine units needed to drive the integrator is not available in the computer. The equation could be solved, however, with a reduced scale factor a_y.

On the basis of these conditions a compromise must be made. It is evident that a choice of the time scale can be made only after a study of the particular problem in question. Some examples will be presented in Chap. 3. In general, a total computing time of 1 to 4 min will be satisfactory for accurate integration. If computer servos are used, they should not be positioned by voltages changing at frequencies higher than 2 to 10 cps, depending on the machine used. If a d-c analog computer is to be used in a *partial system test* (see Sec. 3.7) in connection with other equipment, it must, of course, be operated on a 1 : 1 time scale.

In many problems, the independent variable, t, say, or some function of t, occurs explicitly in the differential equations.

Example.

$$\frac{dy}{dt} = -y + t - t^3 - \sin 2t$$

In such cases, t is not represented by the machine time τ alone. It is necessary to define an additional voltage or machine variable which will represent t in the machine. In accordance with previous practice, this machine variable will be denoted by T and will be defined by a transformation equation

$$T = a_t t \tag{2.5}$$

The scale factor a_t must not be confused with the time-scale factor α_t, from which it is in general numerically and dimensionally different.[1] The machine variable T is necessarily always generated as a function of the machine time τ, which is the real independent variable in the machine. It is, therefore, quite proper to consider T as a dependent and not as an independent machine variable.

2.4. Machine Equations and Block Diagrams. As soon as all scale factors, for the independent variable as well as for the dependent variables, have been chosen (subject to later revisions), the transformation equations (2.1), (2.3), (2.4), and (2.5) should be collected and recorded on a sheet or card clearly labeled with the problem number or designation. Thus,

$$X = a_x x \qquad \Delta = a_\delta \delta \qquad V = a_v v \qquad \cdots \qquad T = a_t t$$
$$\tau = \alpha_t t \qquad P = \frac{1}{\alpha_t} p \tag{2.6}$$

These equations are next solved for the original mathematical variables:

$$x = \frac{1}{a_x} X \qquad \delta = \frac{1}{a_\delta} \Delta \qquad v = \frac{1}{a_v} V \qquad \cdots \qquad t = \frac{1}{a_t} T$$
$$t = \frac{1}{\alpha_t} \tau \qquad p = \alpha_t P \tag{2.7}$$

(2.7) is recorded on the same sheet or card as (2.6).

The given differential equations are, of course, expressed in terms of the mathematical variables $(x,\delta,v, \ldots ,t)$. It is easy to rewrite the given differential equations in terms of the machine variables $(X,\Delta,V, \ldots ,T,\tau)$. It is merely necessary to substitute the expressions (2.7) for the original variables in the given differential equations. The resulting mathematical relations between the machine variables are called the *machine equations;* they are the equations actually solved by the computing machine.

Example. Consider a given set of differential equations

$$\left.\begin{aligned} p^2y + 7.34py + 4.9px + 3.5y + 100x &= 0 \\ p^2x + 15.9py + 10px + 731y + 50x &= 200 \end{aligned}\right\} \tag{2.8}$$

---

[1] The numerical value of a_t is determined by simply applying Eq. (2.2) to the independent variable.

If the scale factors are chosen to be

$$a_x = a_y = \frac{1}{100}\left(\frac{\text{machine units}}{\text{unit}}\right) \qquad \alpha_t = 10\left(\frac{\text{seconds}}{\text{unit}}\right) \tag{2.9}$$

the transformation equations become

$$x = 100X \qquad y = 100Y \qquad t = \tfrac{1}{10}\tau \qquad p = 10P \tag{2.10}$$

These expressions are substituted into the original differential equations. The machine equations are

$$\left.\begin{aligned} P^2X + 1.59PY + PX + 7.31Y + 0.5X &= 0.02 \\ P^2Y + 0.734PY + 0.49PX + 0.035Y + X &= 0 \end{aligned}\right\} \tag{2.11}$$

The machine equations express mathematical relations between *voltages* in a form more or less readily set up on the machine. The arrangements of computing elements which will establish these relationships between the voltages in the computer are first drawn up on paper as *block diagrams*, one for each equation. Each block diagram will serve as a visual aid for connecting the computing elements proper and also constitutes an important record of the computer setup under consideration.

Just as a mathematical equation describes relations between variables and operators, the block diagram describes relations between voltages (machine variables) and computing elements (machine operators). The *basic computing elements* available for this purpose were already discussed in Chap. 1. Some symbols commonly used in block diagrams are shown in Fig. 2.1, together with a summary of the operations performed by each of the basic computing elements. On all block diagrams, the machine variables at all amplifier output terminals should be clearly indicated so that the block diagram can be easily interpreted.[1]

The block-diagram symbols presented in Fig. 2.1 are intended as suggestions only and will frequently have to be supplemented by more complete diagrams (partial circuit diagrams). This is especially true for setups on computers which have no fixed computing and feedback networks built into the machine. Since block diagrams for simple relations are most easily understood, it is best to draw separate block diagrams for each machine equation or even several block diagrams for the various relations expressed by one machine equation.

Each block diagram should be drawn so that it can be "read" from one or more input terminals on the left to one or more output terminals on the right like a sentence or an equation in a book. It is not necessary or desirable to show interconnections between the output and input terminals of the block diagrams corresponding to two different machine equations. Such interconnections are, in any case, clearly seen from the fact that the respective terminals are labeled with the same symbols.

---

[1] In a number of computer laboratories, it is customary to label voltages in block diagrams with the products of scale factors and mathematical variables (such as $a_y y$ instead of Y). It is believed that this procedure will permit errors more easily than that described in the text.

The computer will be set up by using wires or patch cords to interconnect the physical components of the machine in the manner shown by the block diagram. *The code or serial number of each actual computing element must be noted carefully on the block diagram as the interconnections are made, and an accurate record must be made of any changes in the computer setup or in the scale factors used.*

Several examples of simple block diagrams, together with the corresponding machine equations, are shown in Fig. 2.2 which also illustrates a number of ways of combining the basic computing elements to perform various operations.

While the number of possible interconnections in a d-c analog computer is very large, several precautions have to be observed which will be brought out more clearly when the design of the individual computing elements is discussed in Chaps. 4 to 6. A practical limit exists on the amount of amplification and thus on the magnitude of a coefficient which can be obtained in most d-c analog computers. Gains higher than plus or minus 200 obtained by means of cascaded amplifiers usually lead to instability.

Again, most amplifiers or integrators should not be loaded by more than two simple potentiometers without special "booster" connections (see Sec. 5.5).

Another point to be remembered is that voltages like $10Y$, $4V$, etc., and also derivatives like PY, P^2Y (see Fig. 2.4) may overload an amplifier output even if the voltages Y, V, etc., do not. Setups involving such situations must be avoided unless the respective scale factors are reduced accordingly.

The simple potentiometers in the computer are usually *loaded* by amplifier input networks. As a result, the potentiometer dial readings are not accurately equal to the respective potentiometer transfer functions. The subject of potentiometer loading is discussed in detail in Sec. 4.1. It suffices to say here that the loading errors can be corrected accurately by reference to a comparison potentiometer or to the correction charts given in Sec. 4.1. *The potentiometer transfer functions indicated in all block diagrams should be the actual transfer functions, including the effects of loading;* otherwise, the machine equations could not be easily derived from the block diagram.

Because of the very great variety of problems which can be solved by means of d-c analog computers, the technique of setting up a machine is

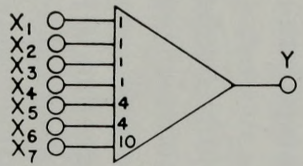
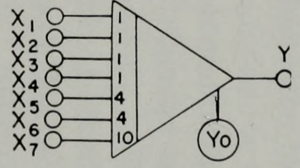
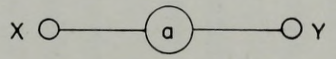
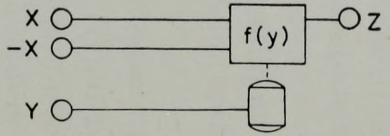
COMPUTING ELEMENTS	BLOCK DIAGRAM SYMBOLS	MACHINE EQUATIONS	OPERATIONS
Summing (Phase Inverting) AMPLIFIER	 X_1 X_2 X_3 X_4 X_5 X_6 X_7 1 1 1 1 4 4 10 Y	$Y = -\{X_1 + X_2 + X_3 + X_4 + 4X_5 + 4X_6 + 10X_7\}$	Summation, Multiplications by constant coefficients with phase inversion
Summing (Phase inverting) INTEGRATOR	 X_1 X_2 X_3 X_4 X_5 X_6 X_7 1 1 1 1 4 4 10 Y Yo	$Y = -\frac{1}{p}\{X_1 + X_2 + X_3 + X_4 + 4X_5 + 4X_6 + 10X_7\}$	Summation, Multiplication by constant coefficients with phase inversion and integration with respect to machine time from initial value Yo
Simple POTENTIOMETER		$Y = aX \quad 0 \leqslant a \leqslant 1$	Multiplication by constant coefficients smaller than 1 in absolute value
Servo-driven function POTENTIOMETER		$Z = Xf(Y)$ $0 \leqslant \lvert f(Y) \rvert \leqslant 1$	Multiplication by variables and functions of variables smaller than 1 in absolute value

FIG. 2.1. Basic computing elements are the building blocks of block diagrams and computer setups.

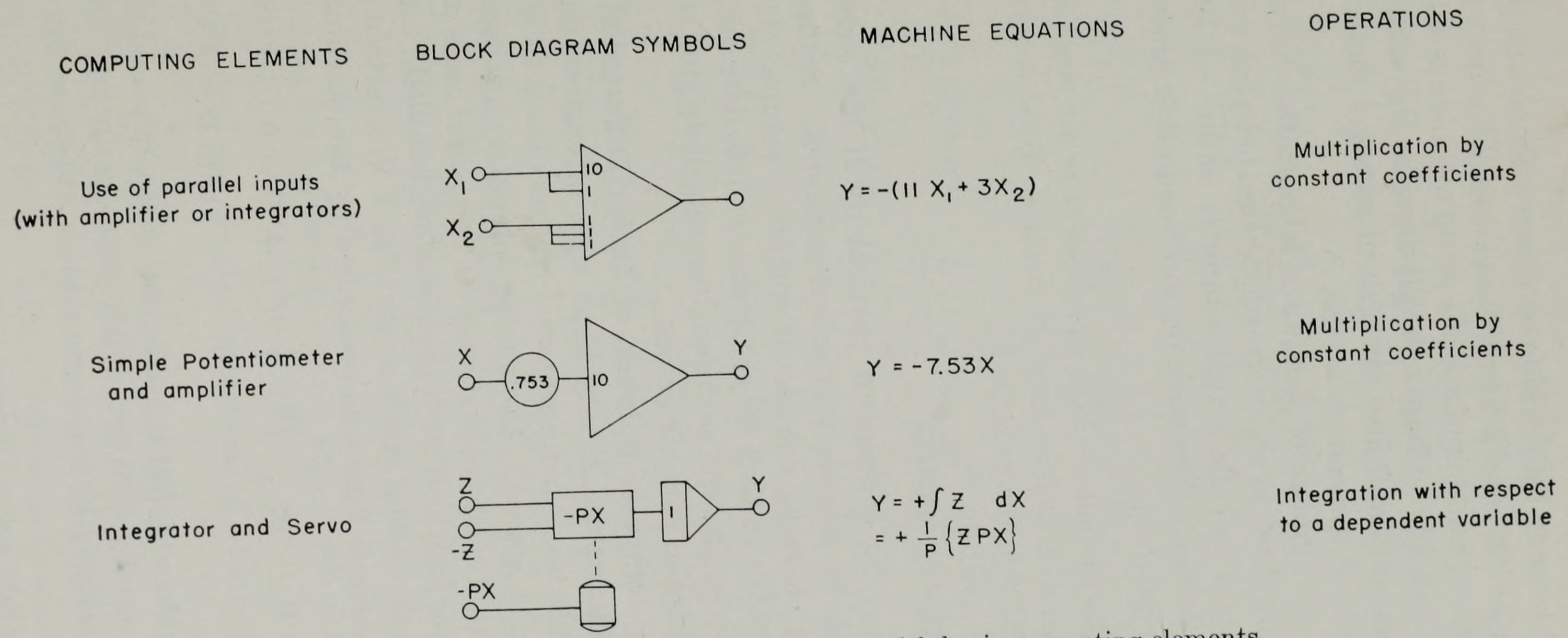

COMPUTING ELEMENTS
BLOCK DIAGRAM SYMBOLS
MACHINE EQUATIONS
OPERATIONS
Use of parallel inputs (with amplifier or integrators)
X_1
X_2
10
1
$Y = -(11X_1 + 3X_2)$
Multiplication by constant coefficients
Simple Potentiometer and amplifier
X
.753
10
Y
$Y = -7.53X$
Multiplication by constant coefficients
Integrator and Servo
Z
-Z
-PX
1
Y
-PX
$Y = +\int Z \quad dX$
$= +\frac{1}{P}\{Z\,PX\}$
Integration with respect to a dependent variable

Fig. 2.2. Examples of combined operations with basic computing elements.

best taught and learned by working as many examples as possible. The material in Secs. 2.5, 2.7, and 2.8 as well as Chap. 3 has been prepared with this idea in mind. A number of specific setup techniques will be discussed in Secs. 2.5, 2.7, and 2.8. Chapter 3 will describe the solution of several complete problems and will also serve to illustrate a method of keeping convenient records of the computations.

Setup Sheets. A number of computer laboratories have successfully used so-called *setup sheets* as an intermediate step between the preparation of the block diagram and the actual computer setup. *A setup sheet is a form listing the following information for each computing element used in the computer setup in question:*

1. *Input machine variables—from which point in the machine*
2. *Output machine variables—to which point(s) in the machine*
3. Mathematical relation enforced
4. Scale factors used
5. Special settings or remarks

Setup sheets may be convenient particularly if the computer setup proper is to be made by relatively unskilled personnel. Setup sheets may also be helpful for checking completed computer setups. On the other hand, it may be difficult to keep the setup sheets up to date during changes in the computer setup, and the additional intermediate step may provide an added opportunity to commit errors.

2.5. D-c Analog-computer Solution of Linear Differential Equations with Constant Coefficients. A very large class of engineering problems concerned with oscillations and automatic controls leads to *sets of linear ordinary differential equations.* D-c analog computers are especially well adapted to the solution of such equations. Two typical cases will serve to illustrate the manner in which such problems are handled.

Suppose that the dependent machine variable Y in a certain problem is to vary with the machine time τ as prescribed by the machine equation

$$\frac{d^2Y}{d\tau^2} + r\frac{dY}{d\tau} + Y = 0 \qquad \text{or} \qquad P^2Y + rPY + Y = 0 \qquad \text{with } 0 \leq r \leq 1 \quad (2.12)$$

It was mentioned already that the differentiations indicated in an equation like (2.12) are usually considered difficult to perform. Differentiations can, however, usually be replaced by integrations, as the following discussion will show.

Equation (2.12) is first solved for the highest derivative of Y, resulting in

$$P^2Y = -(rPY + Y) \qquad (0 \leq r \leq 1) \qquad (2.13)$$

Each of the terms on the left and right of Eq. (2.13) will be a voltage. The voltage P^2Y must appear as a *sum*, multiplied by -1, of the voltages rPY and Y.

This relationship between voltages is obtained by applying the voltages rPY and Y to the input terminals of a summing amplifier (Fig. 2.3*a*) which will perform the required operations of addition and multiplication by -1.

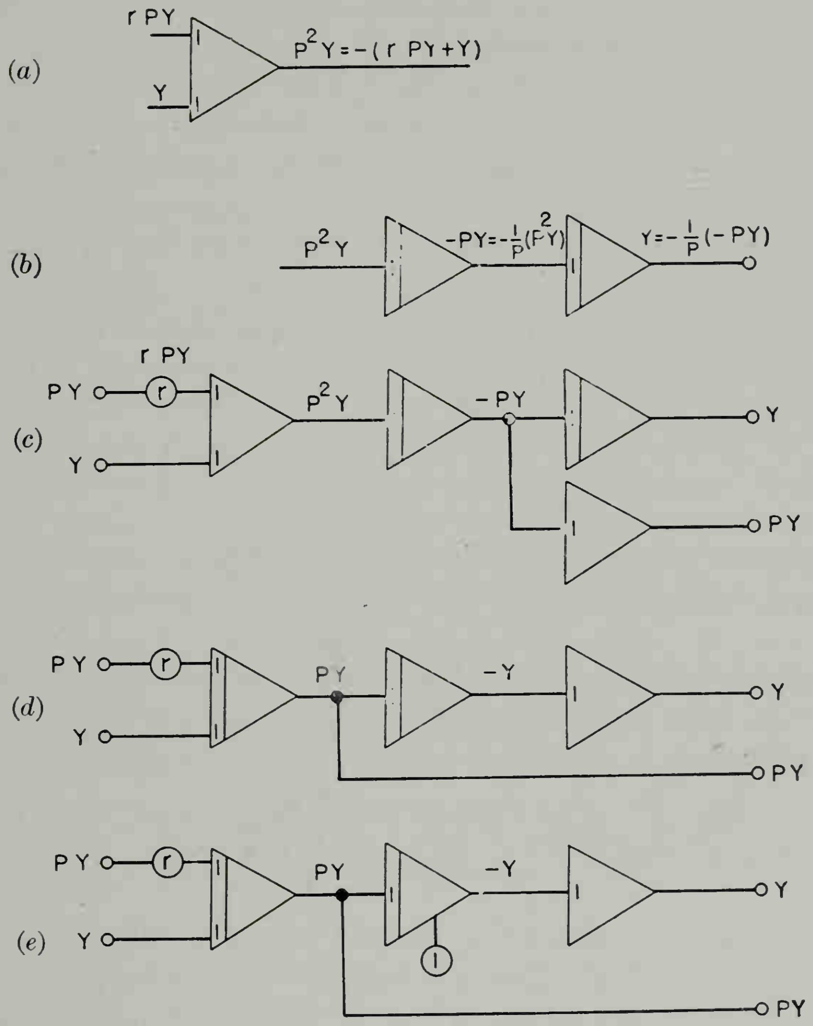


FIG. 2.3. Solution of the differential equation $P^2Y + rPY + Y = 0$ with $0 \leq r \leq 1$. Initial conditions: $Y_0 = 1$, $(PY)_0 = 0$.

The voltages rPY and Y, which must appear at the summing amplifier input terminals, are not independent of P^2Y but must be obtained from P^2Y through successive integrations and multiplications by constants. The voltage P^2Y is treated as if it were already known and is applied to the input of an *integrator* whose output must then be equal to $-PY$. Integration of this voltage in turn yields the voltage Y; each electronic integration results in a change of sign (Fig. 2.3*b*). The voltage $-PY$ is, finally, multiplied by $-r$ by means of a phase-inverting amplifier and a simple potentiometer, so that the required voltage rPY is obtained. Figure 2.3*c* shows the block diagram corresponding to the machine equation (2.12), except for the initial condition settings. The latter will be discussed in Sec. 6.

Experience has shown that there is usually more than one correct computer setup for any given problem. One or more of these may be preferable to others for reasons of economy in the use of computing elements because of better accuracy or because of scale-factor advantages. For this reason, every block diagram should be inspected with a view to possible revisions. Unless it is necessary to measure or record the voltage P^2Y itself, the operations of summing and integrating in Fig. 2.3*c* may both be performed by a summing integrator. This results in the saving of one amplifier. The *revised block diagram* is shown in Fig. 2.3*d*. A check shows that the revised block diagram corresponds to the relation

$$PY = -\frac{1}{P}\left[rPY - \frac{1}{P}(-PY)\right]$$

which is equivalent to Eq. (2.12). It is seen that *the machine variable Y will have no choice but to vary as prescribed by the given machine equation* (2.12).[1]

As a second example, suppose that two dependent machine variables X and Y are to vary with the machine time τ as prescribed by the set of machine equations derived in the example of Sec. 2.3, *viz.*,

$$\left.\begin{aligned} P^2X + 1.59PY + PX + 7.31Y + 0.5X &= 0.02 \\ P^2Y + 0.734PY + 0.49PX + 0.035Y + X &= 0 \end{aligned}\right\} \qquad (2.14)$$

In such problems, one machine equation is solved for the highest occurring derivative of each of the machine variables. This procedure is applicable even if the highest derivatives of two or more machine variables do not occur in more than one of the equations. It is always possible to differentiate one of the other equations until the desired derivative is obtained.

In the present case, one of the machine equations (2.14) is solved for each of the variables P^2X and P^2Y:

---

[1] Equation (2.12) is the equation of damped harmonic motion. If $Y_0 = 1$, $(PY)_0 = 0$, the solution will be found to be

$$Y = Y_0e^{-b\tau}\cos\omega\tau \qquad b = \frac{r}{2} \qquad \omega = \sqrt{1 - \frac{r^2}{4}}$$

This solution is easily recorded and verified for a number of values of r. For this reason, this simple problem is often used to check a new computer.

$$\left.\begin{aligned} P^2X &= -(1.59PY + PX + 7.31Y + 0.5X - 0.02) \\ P^2Y &= -(0.734PY + 0.49PX + 0.035Y + X) \end{aligned}\right\} \qquad (2.15)$$

Once the desired relations between the voltages in the machine have been expressed in the form (2.15), they can be easily established physically. Each Eq. (2.15) is taken up in turn.

Each of the terms in (2.15) will be a voltage or machine variable. Starting with the first equation (2.15), the voltage P^2X must appear as the *sum*, multiplied by -1, of the voltages

$$1.59PY,\ PX,\ 7.31Y,\ 0.5X,\ -0.02 \text{ machine unit} \qquad (2.16)$$

This end is achieved by applying the voltages

$$\frac{1.59}{4}PY,\ PX,\ \frac{7.31}{10}Y,\ 0.5X,\ -0.02 \text{ machine unit} \qquad (2.17)$$

to the correct input terminals of a *summing amplifier* (see Fig. 2.4*a*), which now performs the required multiplications by -1, -4, and -10 as well as the addition. The voltages (2.17) are not independent of the voltages P^2X and P^2Y but must be obtained from them through successive integrations and multiplications by constants. As before, the voltage P^2X is treated as if it were already known and is applied to the input of an *integrator* whose output must then be equal to $-PX$. In this manner, *successive integrations* by means of cascaded integrators (Fig. 2.4*b*) yield the voltages $-PX$ and X at the output terminals of successive integrators. Similarly, successive integrations of P^2Y yield the voltages $-PY$ and Y. Each electronic integration results in a change of sign. It should be noted that an integrator may be used to effect a multiplication by a constant factor as well as an integration.

It remains to *multiply* the voltages $-PX$, X, $-PY$, and Y by the proper positive and negative constants by means of phase-inverting *amplifiers* and *potentiometers* to obtain the voltages (2.17) which are the terms to be summed by the first summing amplifier.

This completes the simulation of the first equation (2.15). A block diagram for the second equation is obtained in essentially the same manner. The machine variable P^2Y appears at the output of a second summing amplifier which sums and inverts the voltages $0.734PY$, $0.49PX$, $0.035Y$, X (machine units), obtained by successive integrations and multiplications of P^2X and P^2Y, as before. A complete, *composite* block diagram corresponding to the machine equations (2.15) is shown in Fig. 2.4*c*.

The composite block diagram should be reexamined with a view to possible improvements in the manner discussed further above.

Figure 2.4*c* shows the complete computer setup except for the initial condition settings. The latter are discussed in Sec. 2.6, and a composite block diagram with the initial condition settings indicated is shown in Fig. 2.5*a*.

Several remarks on such block diagrams in general are in order. It is seen that *a separate block diagram has been drawn for each of the machine equations.* Each machine variable appears at an output terminal on the right of one of the equation block diagrams and, usually, at one or more of the input terminals on the left. Several other voltages (auxiliary machine variables) also appear at the right and left and may or may not be recorded. The only input terminals will be those connected to a source of constant voltage or an external function generator "driving" the computer. *The interconnections between terminals on the right and left or the interconnections between block diagrams of different equations need not and should not be drawn.* This procedure will result in much greater clarity of the composite block diagram and will also aid in the handling of the records. This fact is illustrated by Fig. 2.5*b*, which shows the block

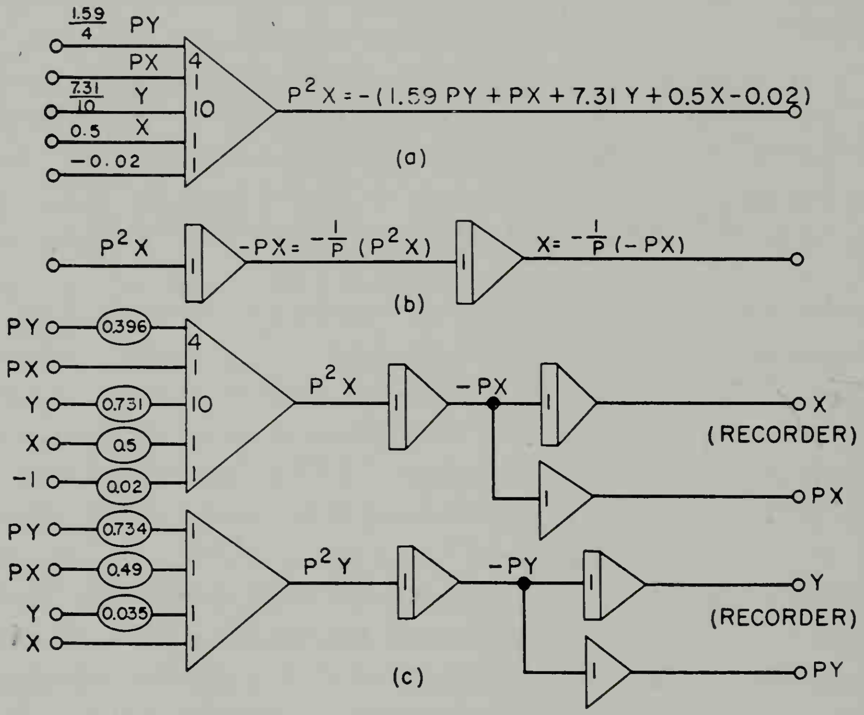


FIG. 2.4. Solution of the set of differential equations

$$P^2X + 1.59PY + PX + 7.31Y + 0.5X = 0.02$$
$$P^2Y + 0.734PY + 0.49PX + 0.035Y + X = 0$$

It is interesting to note in the above computer setup how all multiplications by factors greater than one have been performed at amplifier inputs. The voltages $1.59PY$ and $7.31Y$ do not occur at amplifier output terminals where they might cause overloading.

diagram of Fig. 2.4*a* with the interconnections drawn in. It is readily seen that the arrangement in Fig. 2.4*a* is more easily interpreted and also permits changes to be made in one equation block diagram without making it necessary to redraw the entire figure.

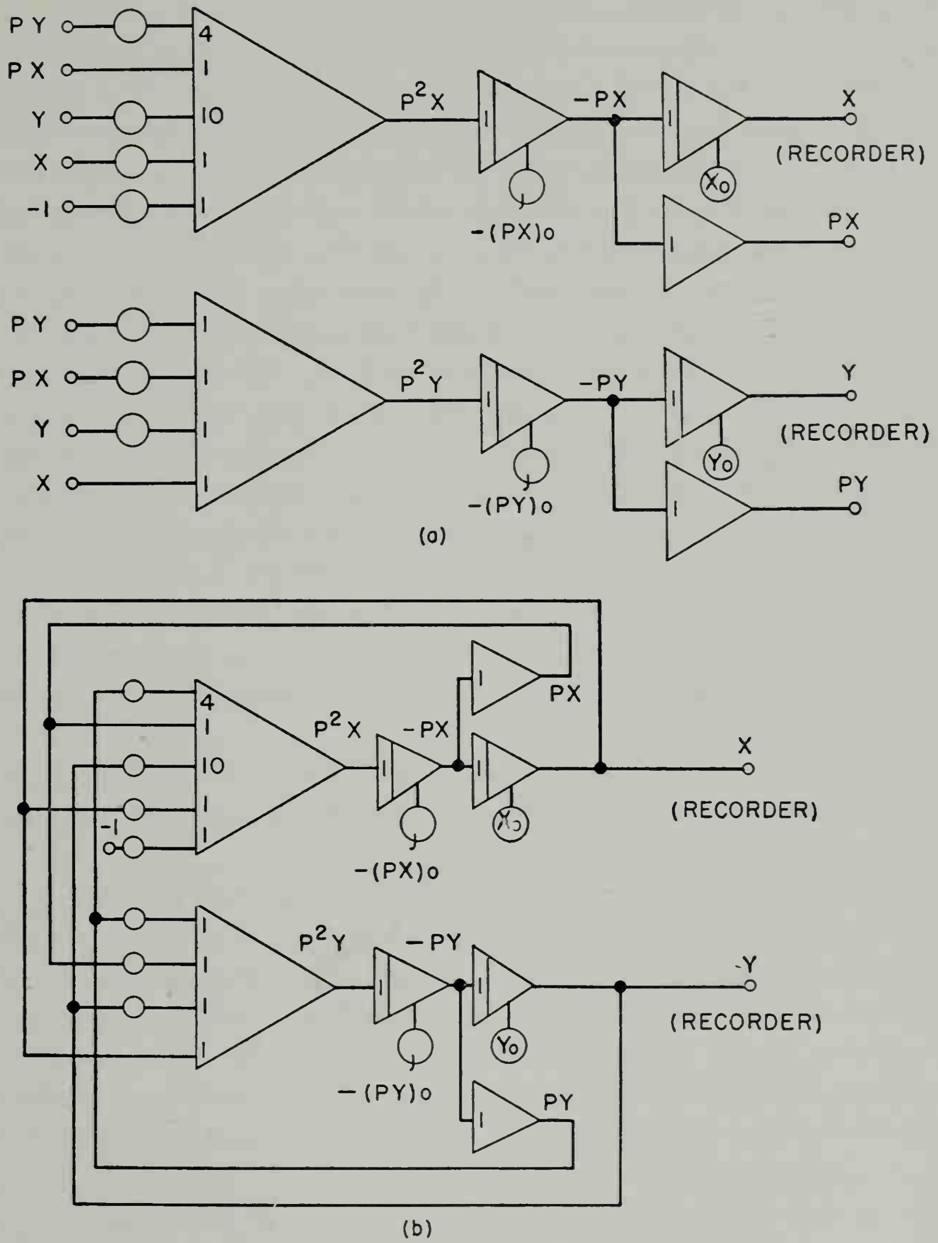


FIG. 2.5. Solution of the set of differential equations

$$P^2X + 1.59PY + PX + 7.31Y + 0.5X = 0.02$$
$$P^2Y + 0.734PY + 0.49PX + 0.035Y + X = 0$$

Whenever a composite block diagram has been established, it should be carefully checked against the corresponding equations. An inspection of Fig. 2.4*a* shows that the machine variables X and Y will have to vary with time as prescribed by Eqs. (2.15) and thus by Eqs. (2.14). Records of the voltages X and Y will, then, constitute a solution of the problem in question.

2.6. Initial Conditions. Before each computer "run" is started, it is necessary to establish at the output of each integrator in the computer a voltage equal to the given initial value of the machine variable which is to appear at that point. The machine variables will then be certain to start to vary from the correct initial values given by the problem.

Naturally, the machine variables could not in general be maintained constant at their initial values before each run while the computer is set up to solve a differential equation. Accordingly, the computer is made inoperative by disconnecting the integrator amplifier grids from the input networks while the initial conditions are set up. This is called the RESET condition of the computer. At the beginning of a computer run, a relay system connects the amplifier grids to the input networks and at the same time removes the initial-condition voltage sources. All integrating capacitors are now charged up to the correct voltages, and the machine variables are free to vary according to the differential equations set up. The exact circuits for the relay system and the initial-condition voltage sources are described in Sec. 7.1. An alternative system, more suitable for introducing initial conditions into *repetitive computers*, is described in Sec. 8.5.

The number of initial conditions will usually equal the number of integrators used for the problem in question, and each integrator output must be set to the corresponding initial-condition voltage.

Summing amplifiers and phase inverters need not be made inoperative in the RESET condition; their output voltages will, in any case, assume the correct initial values corresponding to the initial values of the voltages established at the amplifier input terminals by preceding integrators.

The correct initial values of the machine variables are easily found by substituting the given initial values of the original problem variables into the transformation equations. In this manner, initial values for the machine variables and their derivatives will be found. It happens frequently, however, that an integrator output voltage, which must be set to the correct initial value, is not simply equal to a machine variable or its derivative. The output of a summing integrator may well be a function of the machine variables and their derivatives. In such cases the correct initial value of this function must be computed and set into the integrator.

Example. Consider a summing integrator performing the operation

$$-\frac{1}{P}(-P^3Y + 4PX)$$

Its output voltage will be $P^2Y - 4X$. Given the initial conditions $(P^2Y)_0 = 0.5$, $X_0 = 0.25$, the correct initial condition setting is $0.5 - 1 = -0.5$.

Figure 2.1 shows the manner in which initial conditions are specified in block diagrams. Figures 2.3*e* and 2.5 show how initial conditions are introduced in the examples of Sec. 2.5. Whenever an integrator is shown without any initial-condition input, the initial value of the integrator output voltage is simply zero.

2.7. D-c Analog-computer Solution of Linear Ordinary Differential Equations with Variable Coefficients. *Introduction.* D-c analog-computer setups for the solution of linear differential equations with variable coefficients and of nonlinear differential equations will require the use of computing elements capable of multiplying two machine variables and of generating functions of machine variables (see Chap. 6). Such computing elements will appear in addition to the "linear" computing elements used for the solution of linear differential equations with constant coefficients. No particular difficulties will be encountered once the limitations on the input and output voltages of the various devices used for multiplication and function generation are understood. Specifically, since the value of a function of a machine variable might exceed the dynamic range allowable in the computer even though the machine variable itself is well within that range, it is often necessary to rewrite the machine equation in the manner shown in Sec. 2.4 so as to keep the output of a function generator within its working range throughout a computation.

Note again that all initial choices of scale factors are preliminary in nature and that the scale factors, including the time scale, may have to be readjusted after a few trial runs in order to obtain the greatest possible accuracy and convenience in each case.

Solution of the Mathieu Equation. Solutions of Mathieu's differential equation

$$\frac{d^2y}{dt^2} + \omega_0^2(1 + \epsilon \cos \omega_m t)y = 0 \tag{2.18}$$

where ϵ, ω_0, and ω_m are constant parameters, are of interest in connection with the study of frequency-modulated oscillations. Inspection of Eq. (2.18) shows that Mathieu's equation is essentially the performance equation of a harmonic oscillator having a "stiffness coefficient" containing a time-variable term.

Assuming that

$$0 < \omega_0 \leq 50 \qquad 0 < \omega_m \leq 100 \qquad 0 < \epsilon \leq 1 \tag{2.19}$$

in Eq. (2.18) and that solutions will be of interest for $0 \leq t \leq 15$, the independent variable may be related to the machine time τ by the transformation equation

$$t = \frac{\tau}{10} \tag{2.20}$$

which leads to the relation

$$\frac{d}{dt} = 10P \tag{2.21}$$

The dependent variable y may be represented in the computer by a voltage of Y machine units such that

$$y = Y \tag{2.22}$$

The scale factor a_y is thus equal to 1, but this figure can be easily raised if an overload should occur in the computer for large values of the initial condition settings.

By substituting the transformation equations (2.21) and (2.22) into Eq. (2.18), one obtains the machine equation

$$100P^2Y = -\omega_0^2\left(1 + \epsilon \cos \frac{\omega_m}{10}\tau\right) Y \tag{2.23}$$

Since, according to (2.19), the value of ω_0^2 may be as large as 2,500, it is convenient to rewrite the machine equation (2.23) in the form

$$\frac{1}{25} P^2Y = -\frac{\omega_0^2}{2{,}500}\left(1 + \epsilon \cos \frac{\omega_m}{10}\tau\right) Y \tag{2.24}$$

so that a coefficient setting potentiometer can be used for multiplication by $\omega_0^2/2{,}500$. The block diagram of Fig. 2.6a shows how a voltage equal to the expression on the right of Eq. (2.23) is formed by means of a summing amplifier having the input voltages $\omega_0^2/2{,}500$ and $(\omega_0^2/2{,}500)\,\epsilon \cos(\omega_m\tau/10)$. A cosine potentiometer capable of continuous 360 deg rotation (see Secs. 6.5 and 6.8) is driven by a clock motor at a speed $\omega_m/20\pi$ rps; this speed is adjusted in accordance with the "modulation circular frequency" ω_m by means of gears.

The machine variables $-\frac{1}{5}PY$ and Y are obtained by successive integrations of the voltage $\frac{1}{25}P^2Y$; the correct initial conditions must be introduced in each integrator in the manner discussed in Sec. 2.6.

Repetitive Computer Setup for Mathieu's Equation. As an interesting alternative, Fig. 2.6b shows the block diagram of a repetitive computer setup for the solution of Mathieu's equation (2.18). Since the repetitive display of the solutions requires the use of a fast time scale (see Secs. 1.1 and 8.5), the transformation equation (2.20) is replaced by

$$t = 10\tau \tag{2.25}$$

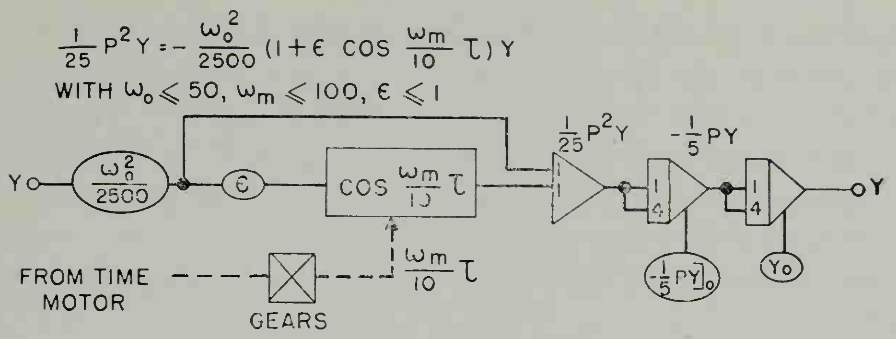


Fig. 2.6*a*. Block diagram of a computer setup for the solution of Mathieu's differential equation (2.18) on a slow time scale. The cosine potentiometer driven by the clock motor is of a type capable of continuous 360-deg rotation (see Sec. 6.5). The potentiometer shaft makes a complete revolution every $\frac{20\pi}{\omega_m}$ seconds. Potentiometer-loading corrections must be observed.

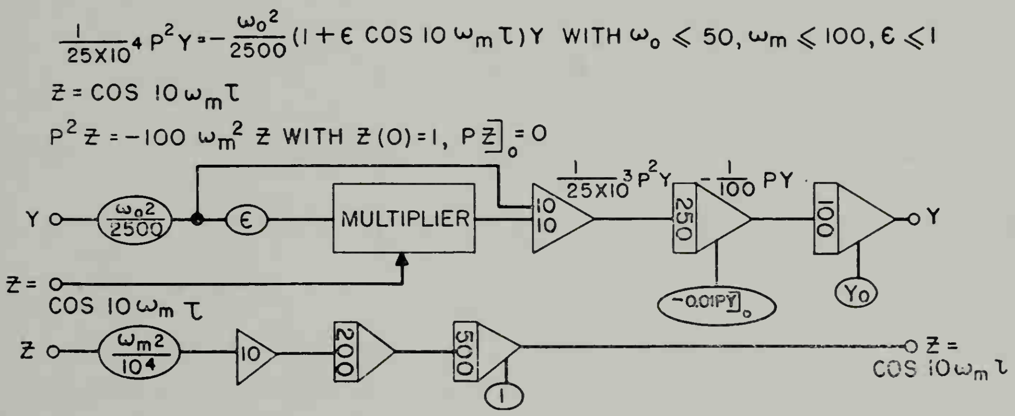


Fig. 2.6*b*. Block diagram for the solution of Mathieu's differential equation (2.18) on a repetitive computer, such as the MacNee machine decribed in Sec. 8.5. The multiplier indicated is a crossed-fields electron beam multiplier (see Sec. 6.1). Note how the function $Z = \cos 10\omega_m\tau$ is generated by implicit solution of the differential equation $P^2Z = -100\omega_m^2 Z$. Integrating capacitors of 0.1 μf are used.

The machine equation becomes

$$\frac{1}{25 \times 10^4} P^2 Y = -\frac{\omega_0^2}{2{,}500} (1 + \epsilon \cos 10\omega_m\tau) Y \tag{2.26}$$

The computer setup of Fig. 2.6*b* is similar in principle to that for the "slow" d-c analog computer (Fig. 2.6*a*), but an electronic multiplier is used to multiply ϵ by a voltage

$$Z = \cos 10\omega_m\tau \tag{2.27}$$

This latter voltage is obtained as a function of the machine time τ as the solution of the differential equation

$$P^2 Z = -100\omega_m^2 Z \qquad \text{where } Z(0) = 1 \text{ and } PZ]_0 = 0 \tag{2.28}$$

This technique of obtaining functions of the machine time τ through implicit computation (see also Sec. 6.5) is widely usable, and its application is by no means restricted to repetitive computers. It will be an interesting exercise for the reader to modify the setup of Fig. 2.6*a* so as to apply this technique. The resulting computer setup is particularly useful if function potentiometers are not available.

Solution of Legendre's Equation. Solutions of Legendre's differential equation

$$(1 - t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + n(n + 1)y = 0 \tag{2.29}$$

where n is a real integer, are important for solving problems in potential theory and antenna theory in terms of spherical coordinates.[1] The initial conditions chosen for n between 0 and 6 are given by the following table:

$$\left.\begin{array}{rccccccc} n & = 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ y(0) & = 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 & -\frac{5}{16} \\ \left.\frac{dy}{dt}\right]_0 & = 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{15}{8} & 0 \end{array}\right\} \tag{2.30}$$

This choice of initial conditions will result in[1]

$$y(1) = 1 \tag{2.31}$$

It will be convenient to represent the independent variable t in the computer by both the machine time τ *and* a voltage or shaft displacement T (see also Sec. 2.3). The dependent variable y will be represented by a machine variable Y.

Assuming that the solutions are of interest only for

$$0 \leq t \leq 1 \qquad -1 \leq y \leq 1 \tag{2.32}$$

the problem variables may be related to the corresponding machine variables by means of the transformation equations

$$\left.\begin{array}{ll} y = \dfrac{1}{a_y} Y & t = T \\ t = \dfrac{\tau}{5} & \dfrac{d}{dt} = 5P \end{array}\right\} \tag{2.33}$$

The scale factor a_y of the dependent variable y is left undetermined for the time being, since estimates for the maximum value of dy/dt may not be available. The machine equation becomes

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[1] For a discussion of the solutions of Legendre's equation in terms of an eigenvalue problem, see L. Pauling and E. Wilson, *Introduction to Quantum Mechanics*, McGraw-Hill, New York, 1935, or H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry*, Van Nostrand, New York, 1946.

$$25P^2Y = \frac{1}{1 - T^2}\,[10TPY - n(n + 1)Y] \qquad \text{where } T = \frac{\tau}{5} \qquad (2.34)$$

It is seen that the machine equation (2.34) or, in fact, *any machine equation corresponding to a single linear and homogeneous ordinary differential equation (with either constant or variable coefficients) is not affected by the scale factor of the dependent variable. The corresponding computer setup is affected by the choice of this scale factor only as far as the initial condition settings are concerned.* It is, then, particularly convenient to find scale factors like a_y in (2.33) through a trial-and-error process, since the computer setup proper will not be affected by changes in a_y. This is not true in the case of computer setups for nonlinear equations.

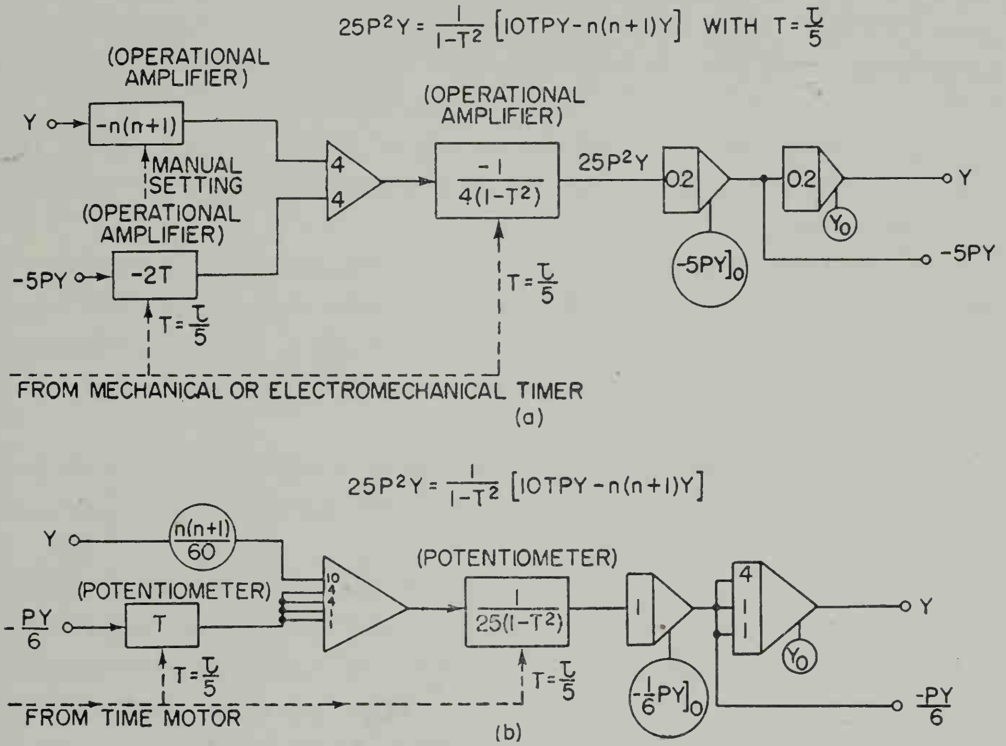

Fig. 2.7. Block diagrams of two d-c analog-computer setups for the solution of Legendre's differential equation (2.29). The function generators indicated in the setup of Fig. 2.7*a*, due to Hagelbarger, Howe, and Howe (see footnote, page 44), are operational amplifiers whose feedback networks are switched by a 40-step tap switch (see Sec. 6.5), in accordance with the desired functions. Figure 2.7*b* is an equivalent CURTIAC setup (see Sec. 8.7) using potentiometer function generators whose transfer functions must be less than unity. Allowances must be made for the effects of potentiometer loading (see Sec. 6.5).

The block diagrams of Fig. 2.7 show two alternative d-c analog-computer setups for the solution of Legendre's equation for values of

n up to 7. In each case, a voltage proportional to P^2Y is obtained as the sum of two voltages corresponding to the two terms on the right of Eq. (2.34). The machine variable Y is then obtained by successive integrations. Initial conditions must be introduced in each integrator as described in Sec. 2.6.

The computer setup of Fig. 2.7*a* is due to Hagelbarger, Howe, and Howe.[1] The machine equation (2.34) is established in a particularly straightforward manner through the use of function generators consisting of phase-inverting amplifiers whose feedback networks are varied stepwise in accordance with the desired function of the time τ by 40-step tap switches (see Sec. 6.5). The transfer functions of such function generators are always negative, but the absolute values of the transfer functions are not limited to the value 1 as in the case of simple potentiometer function generators.

Nevertheless, since the function $1/(1 - T^2)$ increases indefinitely for values of T approaching unity, it is necessary to replace the function by its value for $T = 0.975$ for all larger values of T. This arrangement limits the absolute value of the transfer function of the function generator to a safe value.

The errors due to this limitation as well as those due to the stepwise generation of the functions $2T$ and $1/4(1 - T^2)$ are quite small, as shown by the results below.

Figure 2.7*b* shows a computer setup for the same machine equation based on the use of tapped potentiometer-type function generators driven by a clock motor, such as those used in the CURTIAC computer described in Sec. 8.7. The tapped potentiometer-type universal function generators must be set up with the load connected as described in Sec. 6.6. The function $1/(1 - T^2)$ is replaced by its value for $T = \sqrt{1 - 1/25}$ for all larger values of the argument T, so that the transfer function of the function generator will never exceed unity.

The computer setups of Fig. 2.7 are capable of solving Eq. (2.29) for values of n between 0 and 7. Trial solutions will show that for $n = 7$, the maximum value of the derivative dy/dt will be about 15, so that the scale factor a_y cannot be larger than $1/15$ machine unit per unit for the computer setup of Fig. 2.7*a* and 2 machine units per unit for the computer setup of Fig. 2.7*b*.

Figure 2.8*a* shows a sample computer record for $n = 6$ obtained by Hagelbarger, Howe, and Howe with the computer setup of Fig. 2.7*a*, and Fig. 2.8*b* shows a comparison between the d-c analog-computer solution

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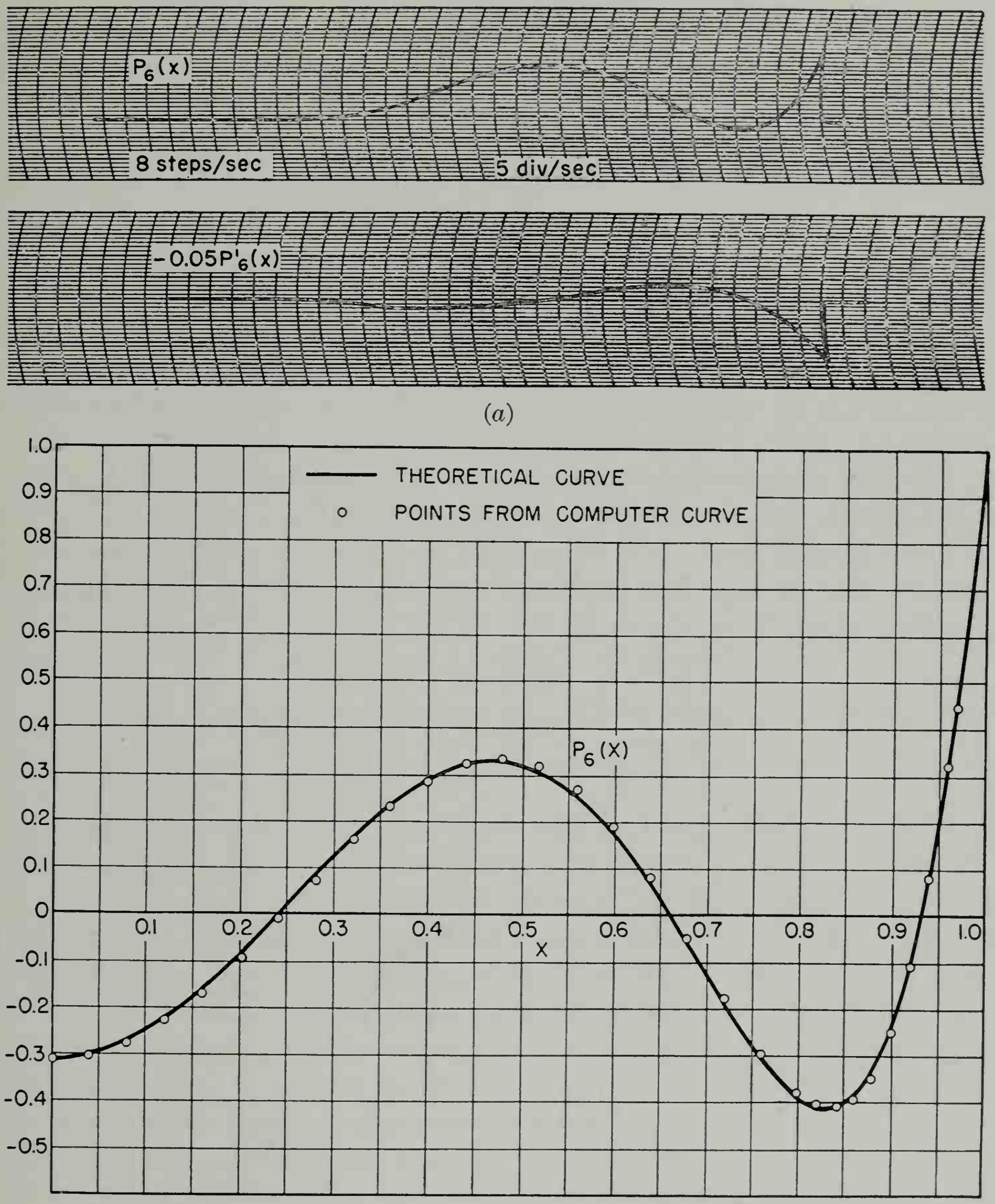
[1] Hagelbarger, D. W., C. E. Howe, and R. M. Howe, *Investigation of the Utility of an Electronic Analog Computer in Engineering Problems*, UMM-28, Engineering Research Institute, University of Michigan, Ann Arbor, Apr. 1, 1949.



(a)

(b)

FIG. 2.8. D-c analog-computer solution of Legendre's differential equation for $n = 6$, and comparison with analytical results (from Hagelbarger, Howe, and Howe, *op. cit.*).

for $n = 6$ and the analytically obtained result

$$y = \tfrac{1}{16}(231t^6 - 315t^4 + 105t^2 - 5) \tag{2.35}$$

for $n = 6$.

2.8. D-c Analog-computer Solution of Nonlinear Ordinary Differential Equations. *The Van der Pol Equation.* The solution of the Van der Pol equation

$$\frac{d^2y}{dt^2} + \mu(y^2 - 1)\frac{dy}{dt} + y = 0 \qquad (2.36)$$

where μ is a constant parameter, may be used, for instance, to describe the build-up of oscillations in certain nonlinear electrical circuits and furnishes an interesting example of the d-c analog-computer solution of a nonlinear differential equation. The Van der Pol equation (2.36) is seen to resemble the performance equation (2.12) of a damped harmonic oscillator but has a nonlinear damping term which will tend to build the amplitude of the oscillations up for small values of y and to decrease the amplitude for large values of y. If an oscillator of this type is given a small initial displacement y, periodic oscillations of constant amplitude will result after an initial transient.

Suppose that $\mu = 0.1$. From previous knowledge or through a trial-and-error process, it may be said that the variable y will not exceed the value 2 appreciably. It is reasonable to relate the problem variables y and t to the corresponding machine variables Y and τ through the transformation equations

$$y = 2.5Y \qquad t = \tau \qquad (2.37)$$

and the resulting machine equation may be written in the form

$$P^2Y = (0.1 - 0.625Y^2)PY - Y \qquad (2.38)$$

The block diagram of Fig. 2.9*a* shows how the machine variable P^2Y is formed through the processes of addition and successive multiplication and how $-PY$ and Y are obtained through integration of P^2Y.

Only linear servo-driven potentiometers are required for the computer setup of Fig. 2.9*a*. A phase-inverting amplifier is used to avoid errors due to excessive loading of the first servo-driven potentiometer. If a function potentiometer is available to perform the required multiplication by Y^2, it is possible to save three operational amplifiers, as shown in Fig. 2.9*b*. Errors due to amplifier drift, distortion, etc., may thus be reduced. Since Y^2 will be less than 1 throughout the computation, it is possible to use the function potentiometer without any necessity for rewriting the machine equation (see Sec. 2.2).

Solution of the Rayleigh Equation. If a new dependent variable z is introduced in Eq. (2.36) through the transformation

$$\frac{dz}{dt} = y \qquad (2.39)$$

one obtains the Rayleigh equation

$$\frac{d^2z}{dt^2} + \mu\left[\frac{1}{3}\left(\frac{dz}{dt}\right)^3 - \frac{dz}{dt}\right] + z = 0 \qquad (2.40)$$

which was originally used for the description of certain acoustical phenomena.

The d-c analog-computer solution of the Rayleigh equation (2.40) is of interest not only for its own sake but will, in view of the transformation (2.39), also furnish the solution of the Van der Pol equation (2.36). This may be particularly useful in view of the fact that the d-c analog-computer solution of Eq. (2.40) requires only the use of a single function generator without the necessity for any multiplications of machine variables. This fact has been used by MacNee[1] for solving either Eq. (2.40) or (2.36) with a comparatively simple setup on his repetitive computer (see Sec. 8.5).

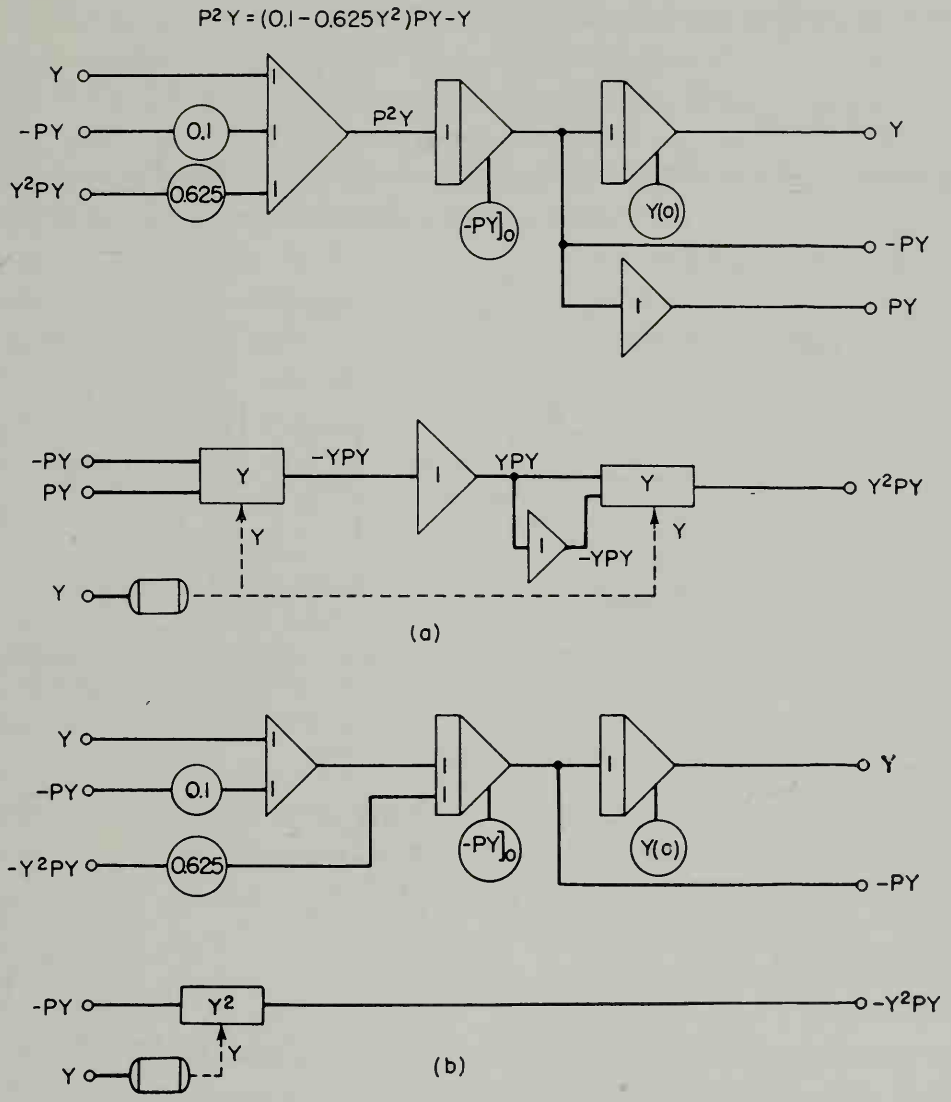


Fig. 2.9. Computer setups for the solution of the Van der Pol equation. Mechanical connections (servo shafts) are indicated as broken lines.

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[1] MacNee, A. B., An Electronic Differential Analyzer, *Proc. IRE*, **37**: 1315, 1949.

Assume again that $\mu = 0.1$. For a repetitive computer solution of Eq. (2.40), the machine variables Z and τ may be introduced through the transformation equations

$$z = 2.5Z \qquad t = 600\tau \tag{2.41}$$

The corresponding machine equation becomes

$$\frac{1}{3.6 \times 10^5} P^2 Z = 0.1 \left(\frac{1}{600} PZ\right) - \frac{0.625}{3} \left(\frac{1}{600} PZ\right)^3 - Z \tag{2.42}$$

Figure 2.10 shows the block diagram of the simple repetitive computer setup for Eq. (2.42); a photoformer (see Sec. 6.5) is used to generate the function $0.208(\frac{1}{600}PZ)^3$.

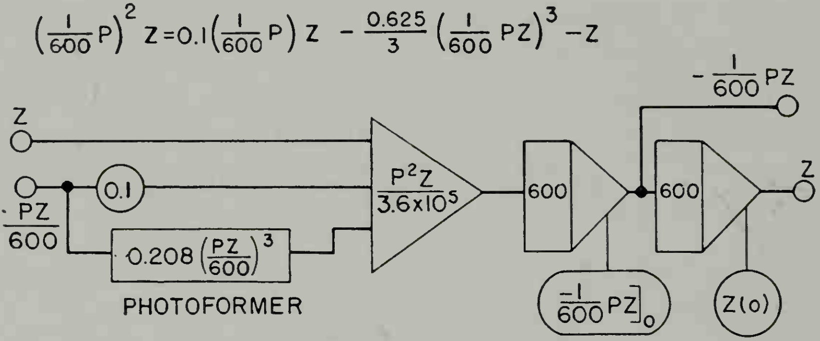
$\left(\frac{1}{600}P\right)^2 Z = 0.1\left(\frac{1}{600}P\right) Z - \frac{0.625}{3}\left(\frac{1}{600}PZ\right)^3 - Z$
$-\frac{1}{600}PZ$
Z
$\frac{P^2Z}{3.6 \times 10^5}$
600
600
Z
0.1
$-\frac{PZ}{600}$
$0.208\left(\frac{PZ}{600}\right)^3$
PHOTOFORMER
$\left[\frac{-1}{600}PZ\right]_0$
Z(o)

Fig. 2.10. Block diagram of a repetitive computer setup for the solution of the Rayleigh equation (2.40). The same computer setup may be used for the solution of the Van der Pol equation (2.36) through the use of the relations (2.45) and (2.46) given in the text. Integrating capacitors of 0.1 μf may be used.

The same computer setup may be used to solve the Van der Pol equation (2.36) through the use of the transformation equations

$$y = 2.5Y \qquad t = 600\tau \tag{2.43}$$

and the relation

$$\tfrac{1}{600}PZ = Y \tag{2.44}$$

corresponding to the transformation (2.39). From (2.42), (2.43), and (2.44) it follows that

$$PZ = 600Y = 240y \tag{2.45}$$

$$\begin{aligned} Z &= 0.1Y - \frac{0.625}{3} Y^3 - \frac{1}{600} PY \\ &= 0.1\left(y - \frac{1}{3} y^3\right) - \frac{dy}{dt} \end{aligned} \tag{2.46}$$

The last two relations will yield the correct initial conditions for Z and PZ from those given for y and dy/dt.

Remarks on the Intelligent Choice of Machine Variables. The use of transformations like (2.39) is often very helpful for simplifying the d-c analog-computer solution of a given problem, especially when only a limited number or choice of computing elements is available.

Again, an intelligent choice of scale factors including the time-scale factor will help one to realize the full accuracy capabilities of the d-c analog computer used. If, for instance, the value of the parameter μ in Eq. (2.36) or (2.40) is increased, the maximum value of the problem variable $z = dy/dt$ will tend to increase without a proportional increase in the maximum value of y. In such cases, the use of a slower time scale is indicated if the accuracy of the computation is to be maintained.

Plots of Solutions in the Phase Plane. The use of servo plotting boards (see Sec. 7.3) for recording d-c analog-computer solutions permits one to plot a problem variable, y, say, against its derivative dy/dt. Studies of such "phase plots" are of particular interest in connection with the theory of nonlinear oscillations.[1] In the case of repetitive computers, phase plots may be obtained conveniently by feeding voltages proportional to Y and PY to the horizontal and vertical deflection plates of a cathode-ray oscillograph.[2]

2.9. Summary of Setup Procedure. The following is a step-by-step review of the setup procedure presented in this chapter. Further examples of applications, together with sample records, will be given in the following sections.

Step 1. *Obtain a complete statement of the problem in question for the record* (*see Sec.* 3.1). *This statement should include, on separate sheets or cards, a clear indication of the object of the study; mathematical relations (differential equations, etc.); initial conditions; numerical values of parameters, given functions, etc.; parameters to be varied; any available solutions or checks; estimated ranges of variables* (*and their derivatives, if possible*).

Step 2. *Determine tentative scale factors for each variable and function and write the corresponding transformation equations on a sheet or card.*

Step 3. *Choose the time scale and write the transformation equation for the independent variable. Using the transformation equations, write the machine equations on another sheet or card. Solve one machine equation for the highest occurring derivative of each machine variable, differentiating some of the equations if necessary. Do not forget to transform initial values, limiting levels, etc., to machine variables.*

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[1] Minorsky, N., *Introduction to Non-Linear Mechanics*, Edwards Bros., Inc., Ann Arbor, Mich., 1947.

[2] MacNee, *op. cit.*

Step 4. *Using separate sheets or cards, draw a computer block diagram for each machine equation. Choose recorder speed and voltage scales.*

Step 5. *Review the combined equation block diagrams. Revise setup and scale factors where necessary to save computing elements, etc. Rewrite equations where necessary.*

Step 6. *Check whether the relations expressed by the revised block diagram reduce to the correct differential equations of the problem.*

Step 7. *Connect the various computing elements by means of the patch cords, following the revised block diagram, equation by equation. Do this with a view to short, convenient connections. Carefully note the number of each actual computing element on the block diagram for reference.*

Step 8. *Set potentiometers, initial values, limiting levels, and function generators according to the block diagram. Correct settings for potentiometer loading where necessary. Set or check recorder calibration.*

Step 9. *Check the entire setup. If at all possible, have a second person check steps* 2 *to* 8 *independently. A good check is also to rederive the machine equations from the computer setup.*

Step 10. *The machine is now ready for trial runs. Revise setup and scale factors where necessary. Keep a careful record of setup and scale-factor changes.*

2.10. An Alternative Method for Handling Scale Factors. *Introduction.* Consider the d-c analog-computer solution of the differential equation

$$\frac{dy}{dt} = -y^2 \qquad \text{or} \qquad py = -y^2 \tag{2.47}$$

According to the procedure described in the foregoing sections, machine variables Y and τ representing the problem variables y and t, respectively, are introduced by the transformation equations

$$y = \frac{1}{a_y} Y \qquad t = \frac{1}{\alpha_t} \tau \tag{2.48}$$

where a preliminary choice of the scale factor a_y is made by observing the inequality

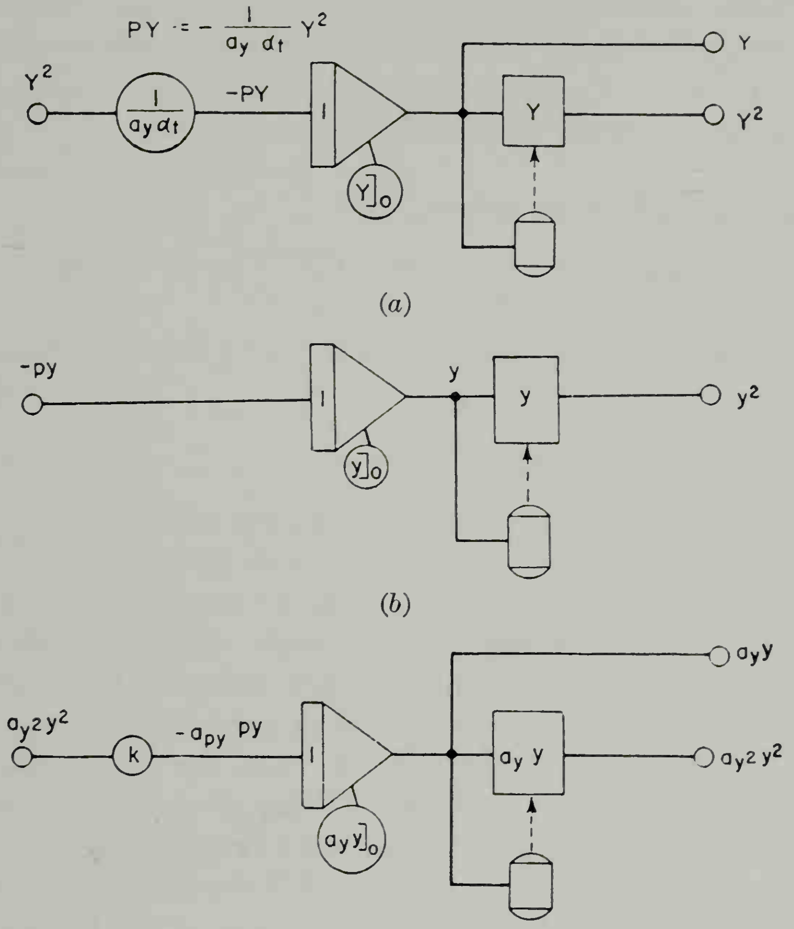
$$Y = a_y y \leq 1 \text{ machine unit} \tag{2.49}$$

in order to avoid overloading the computing elements. The resulting machine equation

$$PY = -\frac{1}{a_y \alpha_t} Y^2 \tag{2.50}$$

can be solved by the computer setup of Fig. 2.11*a* if the time scale chosen is assumed to be sufficiently slow that $1/\alpha_t a_y \leq 1$; if a faster time scale (larger α_t) is desired, the coefficient setting potentiometer shown must be supplemented by a pair of amplifiers.

An Alternative Procedure. The following alternative procedure for handling scale factors avoids the introduction of explicit symbols for the machine variables Y and τ:



(*a*)

(*b*)

(*c*)

FIG. 2.11. Illustrating different methods of handling scale factors in the d-c analog-computer solution of the differential equation $dy/dt = -y^2$. The procedures shown are basically equivalent.

1. A preliminary block diagram of a computer setup for the desired equation is drawn *without regard to scale-factor restriction for the various computing elements* (see Fig. 2.11*b*). In general, the relations expressed by this block diagram cannot be enforced by the actual computing elements, since their dynamic ranges may be too small or too large.

2. A similar block diagram is now drawn (Fig. 2.11*c*) for machine variables obtained by introducing (as yet unknown) scale factors for the voltages appearing at each terminal throughout the block diagram. In the notation chosen, $a_y y$ will be the machine variable representing y,

$a_{y^2}y^2$ that representing y^2, $a_{py}py$ that representing py, etc. A coefficient setting potentiometer (or one or more amplifiers) is provided in each feedback loop for adjusting the scale factors.

3. It is seen that there are more scale factors to be chosen than can be determined by the inequality (2.49). An inspection of the block diagram shows, however, that relations between the scale factors are enforced by the computing elements; thus, in the case of Fig. 2.11*c*, one has

$$a_{py} = \frac{a_y}{\alpha_t} \qquad \text{and} \qquad a_{y^2} = (a_y)^2 \tag{2.51}$$

If the resulting machine equation

$$-a_{py}py = ka_{y^2}y^2 \qquad \text{or} \qquad -\frac{a_y}{\alpha_t}py = k(a_y)^2y^2 \tag{2.52}$$

is to be equivalent to the given equation (2.47), one must have

$$k = \frac{1}{a_y\alpha_t} \tag{2.53}$$

which corresponds to the potentiometer setting in Fig. 2.11*a*.

The so-called *closing equations*[1] (2.51) and (2.53) together with the inequality (2.49) determine the potentiometer setting k as well as *all* the scale factors in Fig. 2.11*c* once the time scale α_t has been chosen; α_t is, again, assumed to satisfy $1/\alpha_t a_y \leq 1$. The values of the potentiometer setting and scale factors obtained in this manner are then entered in the block diagram (Fig. 2.11*c*), which is then used for the computer setup. It goes without saying that, with this method of scale-factor adjustment as with any other, the computer setup may require readjustments after trial runs indicate possible improvements in the choice of scale factors.

Equivalence of the Two Methods. As evidenced by the foregoing discussion, *the two methods for handling scale factors are equivalent in that the computer setups and machine variables resulting from either setup method can be identical with their respective counterparts resulting from the alternative setup procedure.* Specifically, the determination of the potentiometer setting k from the closing equation (2.52) in the example corresponds to the computation of the machine equation (2.50).

For this reason, any of the block diagrams presented in Chaps. 2 and 3 are equivalent to those derived by the alternative setup method if one remembers that the machine variables indicated (such as Y, $6X$, $P\Delta$, etc.) are equal to products of the corresponding problem variables by the respective scale factors (such as $a_y y$, $6a_x x$, $(a_\delta/\alpha_t)\ p\delta$, etc.).

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[1] Caldwell, S. H., and V. Bush, A New Type of Differential Analyzer, *J. Franklin Inst.*, October, 1945, p. 255.

Although no mathematical reasons exist for preferring either one of the setup procedures presented, the methodical introduction of explicitly defined machine variables by a set of transformation equations appears to have certain striking advantages. The authors' experience seems to indicate that this approach makes it easier to understand the operation of the machine, which, after all, handles voltages and shaft rotations rather than multiples of problem variables. The explicitly written transformation equations and machine equations are also particularly helpful in avoiding errors, especially errors associated with the effects of changing the time scale.

CHAPTER 3

THE APPLICATION OF D-C ANALOG COMPUTERS TO REPRESENTATIVE PRACTICAL PROBLEMS

3.1. Problem Preparation. *Introduction: Performance Studies.* D-c analog computers are particularly well suited for the solution of problems involving the solution of sets of ordinary differential equations, and the majority of the sample problems presented in this chapter are of this general type. The solution of problems involving the solution of ordinary transcendental or algebraic equations is also discussed briefly in Sec. 3.9.

Most of the problems discussed in this chapter involve d-c analog-computer studies of the performance of engineering systems. The various performance characteristics of dynamical, electrical, or electromechanical systems can usually be inferred from records of the system coordinates vs. the time. The use of an analog computer permits one to obtain such records quite rapidly as machine solutions of sets of differential equations (equations of motion or performance equations) which specify the design of the system under consideration. Wide ranges of variations in the design of an engineering system under study can be conveniently explored by making appropriate corresponding changes in the computer setup.

Parameter Studies: Design by Trial and Error. Methods involving the systematic substitution of different combinations of unknown design parameters are frequently used to find those "optimal" combinations of design parameters which will be the most useful for achieving a desired result. Trial-and-error methods of this type, made possible through high-speed computing methods, are frequently preferable to more conventional analytical engineering processes in the design of complex systems, such as nonlinear automatic control systems.

Parameter Studies: Analyses Based on Insufficient Data. Quite frequently, the values of parameters important for an analysis cannot be known with precision at the time the analysis is made. In such cases, bracketing of these parameters will permit the investigator to check the sensitivity of the results to changes in the parameters in question; such checks will help to evaluate the validity of results based on assumed approximate values of the parameters. The application of automatic computing machinery will, in fact, make it possible for the investigator to evaluate the effects of many combinations of the unknown parameters.

Preparation of Records. From the point of view of convenience in setting up the computer as well as for reference purposes, it is desirable to record all the information pertinent to the solution of a given problem as clearly and unambiguously as possible. For this purpose, it is useful to prepare two sets of forms, one summarizing the *problem specifications* and one constituting a *record of the computer solution.*

Problem Specifications Forms. The *problem specifications forms*[1] will contain the following information:

1. *Introduction.* This paragraph will contain problem designations, authorization references, etc.
2. *Statement of the problem.* This paragraph will contain a concise statement of the computations to be performed and the specific results desired from the computations. Sketches and diagrams may be included.
3. *Mathematical relations.* All mathematical equations and inequalities relevant to the computations must be listed.
4. *Glossary of symbols.* All symbols used must be defined for rapid reference.
5. *Initial conditions and boundary conditions* (in the case of problems involving differential equations).
6. *Numerical values of constants.* The values of the constant coefficients as well as the units used must be specified.
7. *Estimated ranges of variables.* The information listed in this paragraph is to aid in an intelligent choice of scale factors (see Chap. 2).
8. *Parameters to be varied.* The values of the parameters to be changed or the ranges of parameter changes for use in parameter studies must be specified. The estimated total number of computer solutions to be thus obtained should also be given.
9. *Expected results of parameter variations* (estimates based on previous computations or practical experience). Such data may be helpful for checking computer solutions.
10. *Possible analytical or numerical checks noted by the customer,* including references to the literature.
11. *Use of results.* This (optional) paragraph may contain information as to the end use of the computations specified in the *problem statement;* such information may be useful in deciding on the representation of the results or in suggesting modifications to the problem statement.
12. *Accuracy requirements.*

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[1] This procedure is followed in the computer laboratory of the Reeves Instrument Corporation, New York City.

A number of practical examples of completed problem specifications forms will be presented in the remaining sections of this chapter and may serve as models for other computations.

Records of Computer Solution Forms. The computer laboratory should prepare "*records of computer solution*" forms for each problem as aids in setting up the computer and as records of the computation for later reference (see also Sec. 9.1). These forms will contain the following information:

1. *Transformation equations* (see Chap. 2). This paragraph amounts to a record of the scale factors used. The units used for problem variables and machine variables must be specified. Changes in the scale factors made in the course of parameter studies must be carefully listed.
2. *Machine equations* (see Chap. 2). This paragraph will contain the relations actually established by the computing elements.
3. *Machine initial conditions* (initial conditions in terms of machine units and dial settings).
4. *Block diagrams.*
5. *Discussion.* Any *comments* regarding the solution of the problem or the operation of the computer should be recorded. In particular, it is important for future checks to list any failures of machine components during the computation. Any special setups or new operating methods should be noted for future reference.
6. *Enclosures* (computer records, graphs, tables of results, etc.).

A number of completed "records of the computation" forms for typical computations are included in the following sections and may serve as models for other computations. The reader should remember, in each case, that *the choice of computer setup and scale factors for a given problem is by no means necessarily unique.*

Evaluation of Machine Solutions. In many cases, the performance of an engineering system for a given combination of design parameters is obvious from an inspection of computer records showing the variation with time of one or several system coordinates. More frequently, no such direct interpretation of the computer records is possible, and the latter must be used indirectly to compute *figures of merit* of the system performance. Examples of such figures of merit are averages or maximum and minimum values of variables or their rates of change. In still other problems, it may also be necessary to analyze the system performance in terms of *cross plots* summarizing the results of several machine solutions, such as frequency-response curves (see Secs. 3.2, 3.3, and 3.7).

3.2. D-c Analog-computer Analysis of a Simple Dynamical System: Performance of a Dynamic Vibration Absorber.

Introduction. As an illustrative example of a dynamical system, consider the dynamic vibration absorber system of Fig. 3.1*a*. A mass m_1, restrained by a spring of stiffness k_1 and a dashpot whose damping coefficient is c_1, is made to oscillate in the y direction through the action of a periodic force

$$F(t) = a \sin \omega t \tag{3.1}$$

This situation is representative of many other situations involving the vibrations of machines containing more or less unbalanced components rotating at a circular frequency ω. In order to reduce the vibration of the main mass m_1, an absorber mass m_2 is coupled to m_1 through a spring of stiffness k_{12}. It may be shown[1] that the parameters m_2 and k_{12} of the vibration absorber may be chosen so that the main mass does not vibrate at all; in this case, the absorber mass m_2 will oscillate in such a way that the force it exerts on the main mass m_1 will just cancel the applied force F. These statements apply only if the force F is of the form (3.1).

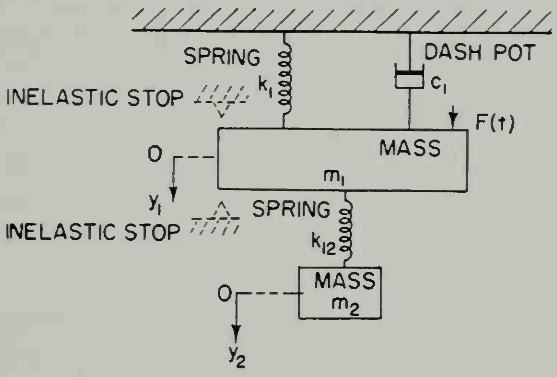

Fig. 3.1*a*. Dynamic vibration absorber system.

Referring to Fig. 3.1*a*, let y_1 be the displacement in the y direction of the main mass m_1 and let y_2 be the displacement of the absorber mass m_2. The equation of motion of the main mass is

$$m_1 \frac{d^2y_1}{dt^2} = -k_1y_1 - c_1 \frac{dy_1}{dt} - k_{12}(y_1 - y_2) + a \sin \omega t,$$

where t is the time. The expression on the right is the sum of the forces acting on the main mass m_1, *viz.*, the restraining force of the main spring, the viscous damping force due to the dashpot, the force exerted by the coupling spring, and the driving force.

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[1] Den Hartog, J. P., *Mechanical Vibrations*, 3d ed., McGraw-Hill, New York, 1947.

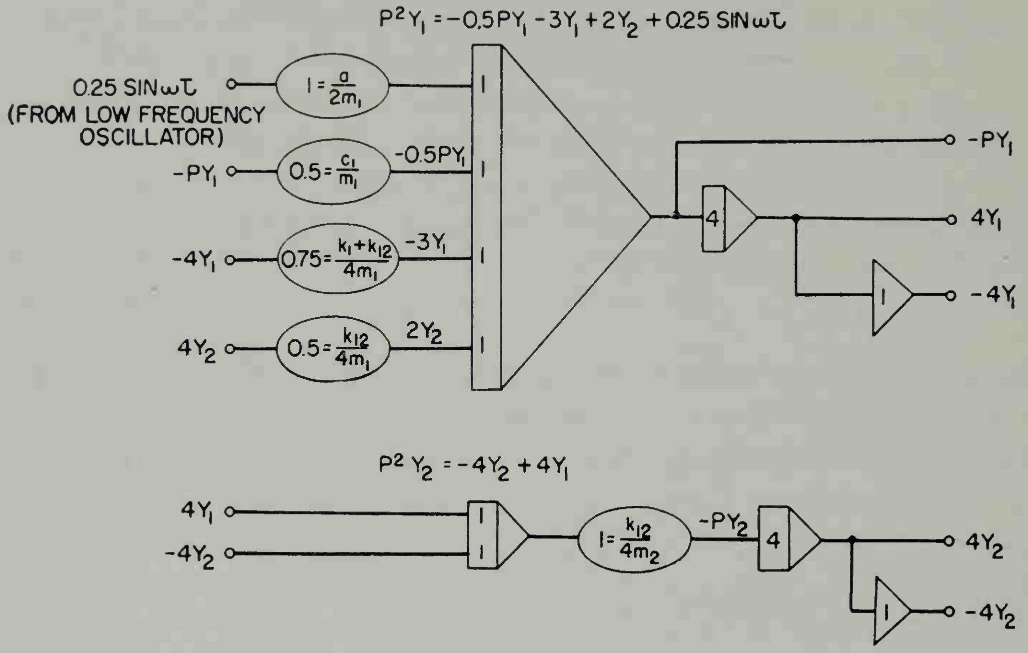

FIG. 3.1*b*. Block diagram of a d-c analog-computer setup for the study of a dynamic vibration absorber system. The coefficients set up on each potentiometer are indicated; the numerical potentiometer settings refer to the sample problem treated in the text.

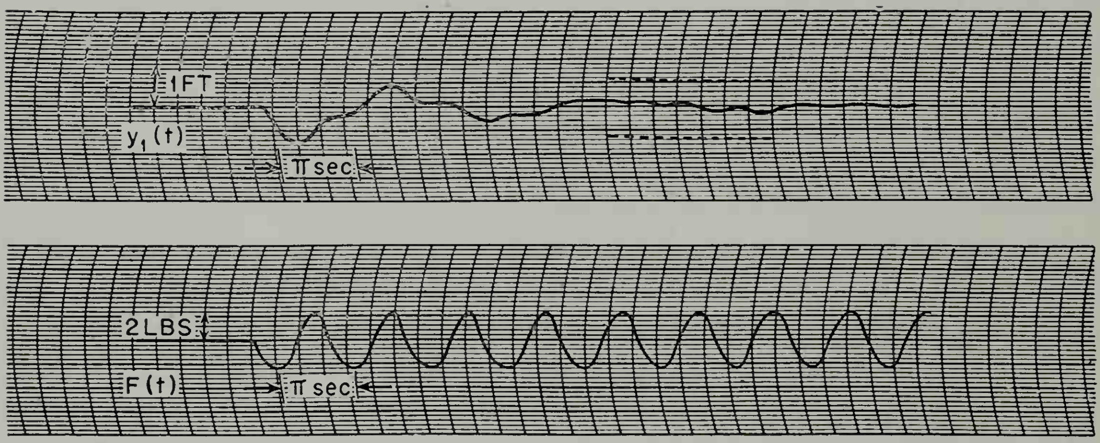

FIG. 3.1*c*. Computer record (Hagelbarger, Howe, and Howe, *op. cit.*), showing the effect of the dynamic vibration absorber near resonance (ω = 2 rad/sec). After an initial transient, the amplitude of y_1 is reduced to less than $1/50$ of the amplitude which would result without the vibration absorber (indicated in the broken lines). The motion of m_1 without the vibration absorber can be studied by setting $k_{12} = 0$ in the block diagram of Fig. 3.1*b*.

Similarly, the equation of motion of the absorber mass m_2 is found to be

$$m_2 \frac{d^2 y_2}{dt^2} = k_{12}(y_1 - y_2)$$

The only force acting on m_2 is that due to the coupling spring.

A practical procedure for handling the study of such a dynamic vibration absorber system in a d-c analog-computer laboratory is illustrated below in terms of a sample *problem statement* and a set of sample *computer setup records.*

Study of a Dynamic Vibration Absorber

A. Problem specifications

1. *Introduction.* This paragraph will contain problem designations, authorization references, etc.
2. *Statement of the problem.* Determine the effect of a dynamic vibration absorber on the motion of a mass (see Fig. 3.1*a*) driven by a force varying as a sinusoidal function of the time. In particular, show how the oscillation amplitude of the main mass is affected by changes in the frequency of the driving force.
3. *Mathematical relations*

$$(a)\quad m_1 \frac{d^2y_1}{dt^2} + c_1 \frac{dy_1}{dt} + k_1y_1 + k_{12}(y_1 - y_2) = a \sin \omega t \qquad \text{(Motion of } m_1\text{)}$$

$$(b)\quad m_2 \frac{d^2y_2}{dt^2} + k_{12}(y_2 - y_1) = 0 \qquad \text{(Motion of } m_2\text{)}$$

4. *Glossary of symbols* (refer to Fig. 3.1*a*)

t	time
y_1	displacement of main mass
y_2	displacement of absorber mass
m_1	main mass
m_2	absorber mass
k_1	stiffness of main spring
k_{12}	stiffness of coupling spring
c_1	damping coefficient
a	amplitude of driving force
ω	circular frequency of driving force

5. *Initial conditions*

$$y_1 = y_2 = \frac{dy_1}{dt} = \frac{dy_2}{dt} = 0$$

6. *Numerical values of constants*

$m_1 = 1$ slug
$m_2 = 0.5$ slug
$k_1 = 1$ lb/ft
$k_{12} = 2$ lb/ft
$c_1 = 0.5$ lb-sec/ft
$a = 2$ lb

7. *Estimated ranges of variables*

$-2 \text{ ft} < y_1 < 2 \text{ ft} \qquad -2 \text{ ft} < y_2 < 2 \text{ ft} \qquad 0 \leq t < 200 \text{ sec}$

8. *Parameters to be varied.* Vary ω between 0.8 and 4 radians/sec in 10 steps.
9. *Expected results of parameter variations.* The amplitude of the steady-state periodic vibrations of the main mass will have a minimum for a certain circular frequency ω. In this "resonance" condition, the sinusoidal oscillations of the absorber mass will be 180 deg out of phase with respect to the driving force.
10. *Possible analytical or numerical checks noted by the customer.* This particular problem may be treated analytically.[1]
11. *Use of results.* Test problem.
12. *Accuracy required.* 0.5 to 1.5 per cent.

B. Record of computer solution

1. *Transformation equations.* (y_1, y_2, and t are measured in the ft/slug/sec system; all machine variables are measured in machine units. One machine unit equals 50 volts.)

$$y_1 = 8Y_1 \qquad y_2 = 8Y_2 \qquad t = \tau$$

   NOTE: The relatively low scale factors for y_1 and y_2 may be necessary to keep the absolute values of PY_1 and PY_2 within the range of 1 machine unit; this choice of scale factors also makes it possible to represent voltages like $4Y_1$ and $4Y_2$ in the machine. The recorder scales are noted on the records.

2. *Machine equations*
   (*a*) $P^2Y_1 = -0.5PY_1 - 3Y_1 + 2Y_2 + 0.25 \sin \omega\tau$
   (*b*) $P^2Y_2 = -4Y_2 + 4Y_1$
3. *Machine initial conditions*
   $Y_1 = Y_2 = PY_1 = PY_2 = 0$
4. *Block diagram.* See Fig. 3.1*b*. An electronic low-frequency oscillator or a signal generator of the type shown in Fig. 6.34 may be used to generate the function $\sin \omega\tau$. Loading corrections for all coefficient setting potentiometers must be taken into consideration.
5. *Discussion* (of special problems encountered during the computation, tentative conclusions from computer results, etc.) "Resonance" takes place near $\omega = 2$ radians/sec; this checks the theoretically computed value $\sqrt{k_{12}/m_2}$ for the resonance circular frequency.
6. *Enclosure.* Dated and initialed computer records ($\sin \omega t$ vs. t, y_1 vs. t, y_2 vs. t).

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[1] See, for instance, *ibid.*

A sample computer record for $\omega = 2$ radians/sec is shown in Fig. 3.1*c*.

Remarks on D-c Analog-computer Studies of Dynamical Systems in General. The d-c analog-computer treatment of the dynamic vibration absorber system may serve to illustrate a number of facts important in connection with practically all d-c analog computer studies of dynamical systems.

1. A definite group of computing elements may be used to establish a machine equation corresponding to the equation of motion for each "generalized coordinate" of the dynamical system.
2. Generally speaking, the input voltages "driving" such a block will represent (generalized) *forces or accelerations*, and the output voltages will correspond to (generalized) *displacements* and *velocities.*
3. Coupling forces leading to logical interactions between two of the equations of motion will result in corresponding electrical interconnections between two blocks of computing elements.
4. It is usually possible to vary system parameters within reasonable limits and to study the resulting changes in system performance without radical changes in the computer setup.

The analogy existing between components of the dynamical system under consideration and "blocks" of computing elements in analog computers makes the latter particularly well suited for engineering studies of dynamical systems.

Analysis of Dynamical Systems Containing Stops and Nonlinear Springs. The motion of the simple undamped vibration absorber can be analyzed without the use of an electronic computer.[1] Consider, however, the case of a dynamical system similar to that of Fig. 3.1*a*, in which the mass m_1 is restrained by inelastic stops (indicated in dashed lines in Fig. 3.1*a*) so that the absolute value $|y_1|$ of the coordinate y_1 cannot exceed a given value $y_{\max}$. Without the use of a differential analyzer, the motion of such a relatively simple system can be analyzed only by tedious numerical methods. The analog-computer approach to the new problem, on the other hand, is essentially as easy to understand as the problem itself. As the problem variable y_1 in Fig. 3.1*a* is restrained between the limits plus and minus $y_{\max}$, the corresponding machine variable Y_1 must be kept between the limits plus and minus $a_{y_1}y_{\max}$. This may be accomplished by means of a pair of parallel diode limiters (see Sec. 6.7). Figure 3.2*a* shows how the computer setup for the first machine equation in Fig. 3.1*b* might be modified so as to include the limiters.

The insertion loss due to the limiting resistor [see Eq. (6.42)] must be taken into consideration in all such computer setups. Figure 3.2*b* shows how the insertion loss due to the limiting resistor in Fig. 3.2*a* is made up by the operational amplifier following the limiter.

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[1] *Ibid.*

Many other types of nonlinear spring or damping characteristics may be simulated by means of various types of function generators (see Chap. 6); thus in systems containing elastic stops, the spring stiffnesses k are seen to vary as functions of the displacements y_1. Servo-driven function potentiometers or tap switches (Sec. 6.5), relays, or diode switches (Sec. 6.7) can be used to simulate such conditions, which may otherwise be quite difficult to analyze. A number of problems involving nonlinear oscillations are discussed in Sec. 2.8.

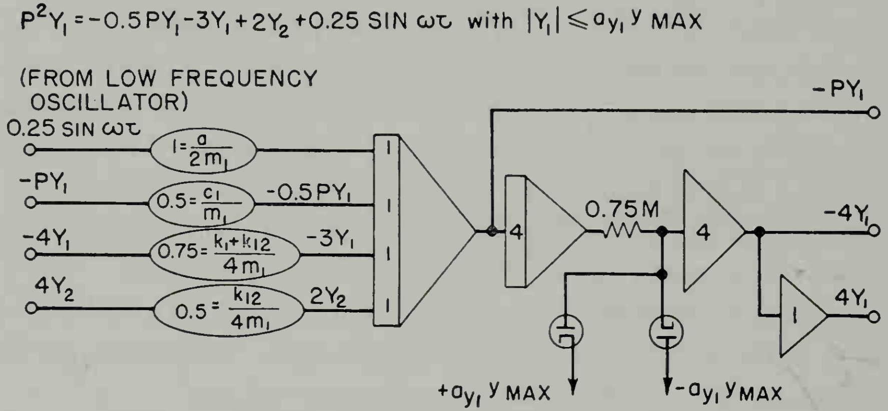
$P^2Y_1 = -0.5PY_1 - 3Y_1 + 2Y_2 + 0.25 \sin \omega\tau$ with $|Y_1| \leq a_{y_1} y_{MAX}$
(FROM LOW FREQUENCY OSCILLATOR)
0.25 SIN ωτ
$1 = \frac{a}{2m_1}$
$-PY_1$
$0.5 = \frac{c_1}{m_1}$
$-0.5PY_1$
$-4Y_1$
$0.75 = \frac{k_1 + k_{12}}{4m_1}$
$-3Y_1$
$4Y_2$
$0.5 = \frac{k_{12}}{4m_1}$
$2Y_2$
1
1
1
1
4
0.75M
4
1
$-PY_1$
$-4Y_1$
$4Y_1$
$+a_{y_1} y_{MAX}$
$-a_{y_1} y_{MAX}$

FIG. 3.2*a*. Possible modification of the computer setup for the first machine equation in Fig. 3.1*b* if Y_1 is limited in absolute value.

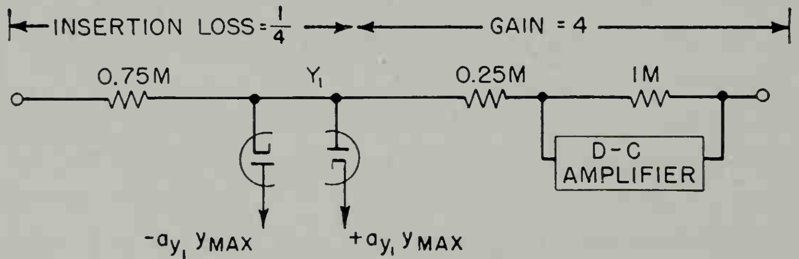

FIG. 3.2*b*. Detail from the computer setup of Fig. 3.2*a*, showing how the insertion loss due to the limiting resistor may be made up by an operational amplifier following the limiter.

The simulation of backlash and hysteresis by means of d-c analog computing elements is discussed in Sec. 3.3.

3.3. Analysis and Synthesis of Servomechanisms by Means of D-c Analog Computers. *Introduction.* Perhaps the most extensive and fruitful applications of d-c analog computers have been in the field of automatic control engineering. D-c analog computing elements lend themselves naturally to the "simulation" of feedback loops analogous to those used in automatic control systems, and the analog nature of the computer input and output data permits one to introduce components of actual control systems into the feedback loops for *partial system tests.*

The following section is intended as an introduction to the analysis and synthesis of servomechanisms by means of d-c analog computers. For an exhaustive treatment of the theory of servomechanisms as such, the reader is referred to the textbooks[1] on this subject.

Performance Equations of a Simple Servomechanism. Figure 3.3*a* shows a simple example of a servomechanism designed to position a load so as to follow the motion of a control dial. A d-c *error voltage e* proportional to the *output error*

$$\epsilon = x_i - x_o \tag{3.2}$$

measured by the potentiometer-type error pickoff device shown is amplified and fed to the d-c servomotor to generate a torque in such a direction as to reduce the error (3.2). The performance equations of this system are determined as follows. The servo amplifier input voltage is

$$e_1 = a_1\epsilon = a_1(x_i - x_o) \tag{3.3}$$

where a_1 is a constant determined by the design of the error pickoff device. The input voltage to the servomotor is

$$e_2 = a_2e_1 = a_1a_2\epsilon = a_1a_2(x_i - x_o) \tag{3.4}$$

where a_2 is the amplifier gain. The equation of motion of the system consisting of the motor, the gear train, and the load is given by

$$(I_L + n^2I_M)\frac{d^2x_o}{dt^2} + n^2r\frac{dx_o}{dt} = na_3e_2 = na_1a_2a_3(x_i - x_o) \tag{3.5}$$

where I_L is the moment of inertia of the load and I_M is the moment of inertia of the motor armature including that of the shaft and gear pinion; the effect of the inertia of the other gears is considered negligible. n is the gear ratio between motor and load, r is the motor damping coefficient, and a_3 is a parameter relating the motor stall torque to the input voltage. The values of r and a_3 may be obtained from the manufacturer's data on the speed-torque characteristics of the particular motor used.[2] The effects of any time delays involved in "building up" the servo torque in the motor mechanism are considered as negligible in this analysis.

D-c Analog-computer Setup for the Performance Equation of the Simple Servomechanism. A d-c analog-computer setup establishing a machine equation corresponding to the servo performance equation (3.5) will now

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[1] Greenwood, I. A., J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948; James, H. M., N. B. Nichols, and R. S. Phillips, *Theory of Servomechanisms*, MIT Radiation Laboratory Series, Vol. 25, McGraw-Hill, New York, 1947; Lauer, H., R. Lesnick, and L. E. Matson, *Servomechanism Fundamentals*, McGraw-Hill, New York, 1947.

[2] See, for instance, Lauer, Lesnick, and Matson, *op. cit.*

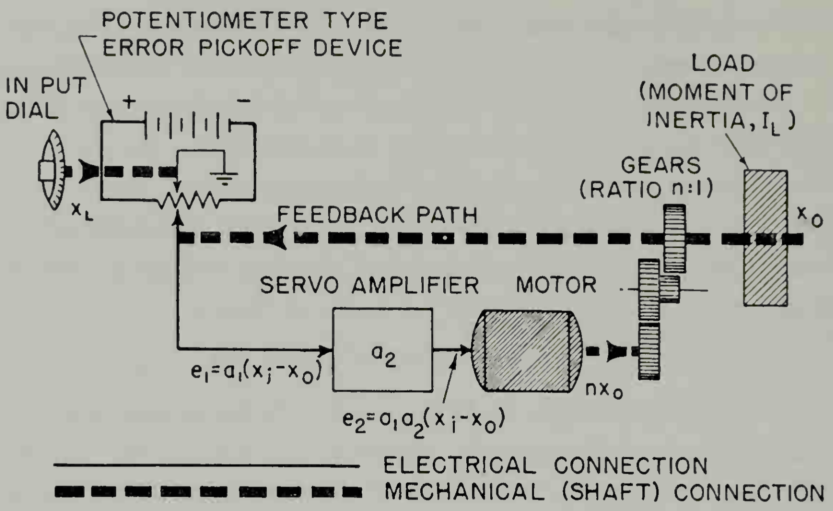

FIG. 3.3*a*. Simple servomechanism designed to make the angular displacement x_o of a load follow the angular displacement x_i of an input dial.

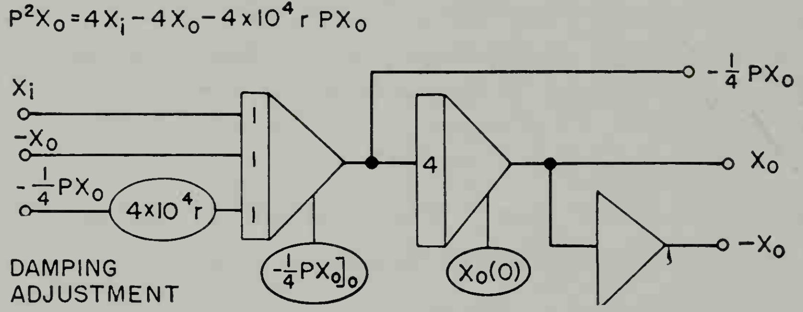

FIG. 3.3*b*. Block diagram of a d-c analog-computer setup "simulating" the servomechanism of Fig. 3.3*a* for the parameter values given in the text.

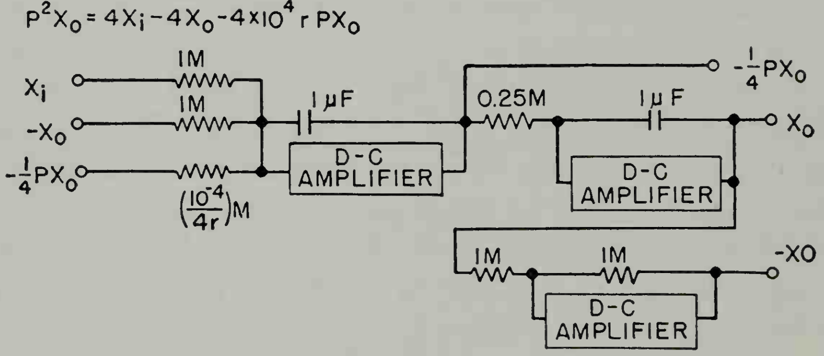

FIG. 3.3*c*. Another computer setup "simulating" the servomechanism of Fig. 3.3*a*. All summing and feedback networks are shown. No coefficient-setting potentiometers are used.

be designed for the following combination of numerical values of the parameters in Eq. (3.5):

$$I_L = 0.05 \text{ slug ft}^2 \qquad I_M = 2 \times 10^{-5} \text{ slug ft}^2$$
$$n = 100 \qquad a_1 = 20 \text{ volts/radian} \qquad a_2 = 20$$
$$a_3 = 2.5 \times 10^{-5} \text{ ft-lb/volt}$$

It is desired to study the effect on the servo performance of varying the motor damping coefficient r between the values

$$r \approx 0 \qquad \text{and} \qquad r = 2 \times 10^{-4} \text{ ft-lb-sec}$$

Physically, the value of r may be changed, for instance, by changing the resistance of the motor armature winding.

If the shaft displacements x_o and x_i are assumed to vary between the limits plus and minus 2 radians, the problem variables t, x_o, and x_i may be related to the corresponding machine variables τ, X_o, and X_i by the transformation equations

$$t = \tau \qquad x_o = 2X_o \qquad x_i = 2X_i \tag{3.6}$$

where the voltages X_o and X_i are measured in machine units of, say, 50 volts. The scale factors chosen are, of course, subject to later revision. The resulting d-c analog-computer solution will be on a 1:1 time scale. The machine equation corresponding to Eq. (3.5) becomes

$$P^2X_o = 4X_i - 4X_o - 4 \times 10^4 rPX_o \qquad \text{with } 0 < r \leq 2 \times 10^{-4} \tag{3.7}$$

Figure 3.3*b* shows the block diagram of a computer setup for the machine equation (3.7). The computer setup is essentially the same as the one for the damped harmonic oscillator discussed in Sec. 2.7.

Figure 3.3*c* shows the block diagram of a similar computer setup more suitable for use with low-cost analog computers (see Sec. 8.4) which have neither coefficient setting potentiometers nor permanently built-in input and feedback networks. Such a computer is set up by connecting resistors and capacitors to the d-c amplifiers in the manner indicated.

D-c Analog-computer Study of the Simple Servomechanism: Transient Response. The computer records of Fig. 3.4 show the response of the simple servomechanism to a step displacement of the input shaft for three values of the damping coefficient r. Parameter studies (see Sec. 3.1) of this type can be made to test the transient response of a servomechanism for any number of combinations of the design parameters; the computer records may be used to pick a combination of design parameters which will result in small errors $x_i - x_o$ without any tendency toward uncontrolled oscillations or hunting. A number of quantitative figures of merit for servomechanism performance will be discussed below.

D-c Analog-computer Setup for a More Complicated Linear Servomechanism. Figure 3.5*a* and *b* shows a semischematic representation and a functional block diagram, respectively, of a somewhat more complex linear servomechanism than that pictured in Fig. 3.3*a*. This servomechanism incorporates a load restrained by springs, rate feedback from a d-c tachometer, as well as special equalization networks which add the rate feedback signal to the error signal and produce both derivative and integral control.[1]

The performance equation of this servomechanism may be written in the form

$$[(I_L + n^2 I_M)p^2 + n^2 rp + k]x_o = n[a_1(x_i - x_o) - nk_1 px_o]a_2 a_3 F(p) \quad (3.8)$$

where k is the stiffness of the spring load, k_1 is the tachometer sensitivity, and $F(p)$ is the transfer function of the equalizing networks, which is here taken to be the same for both the position error and the rate feedback signals. The other symbols are the same as those used in the case of the simple servomechanism.

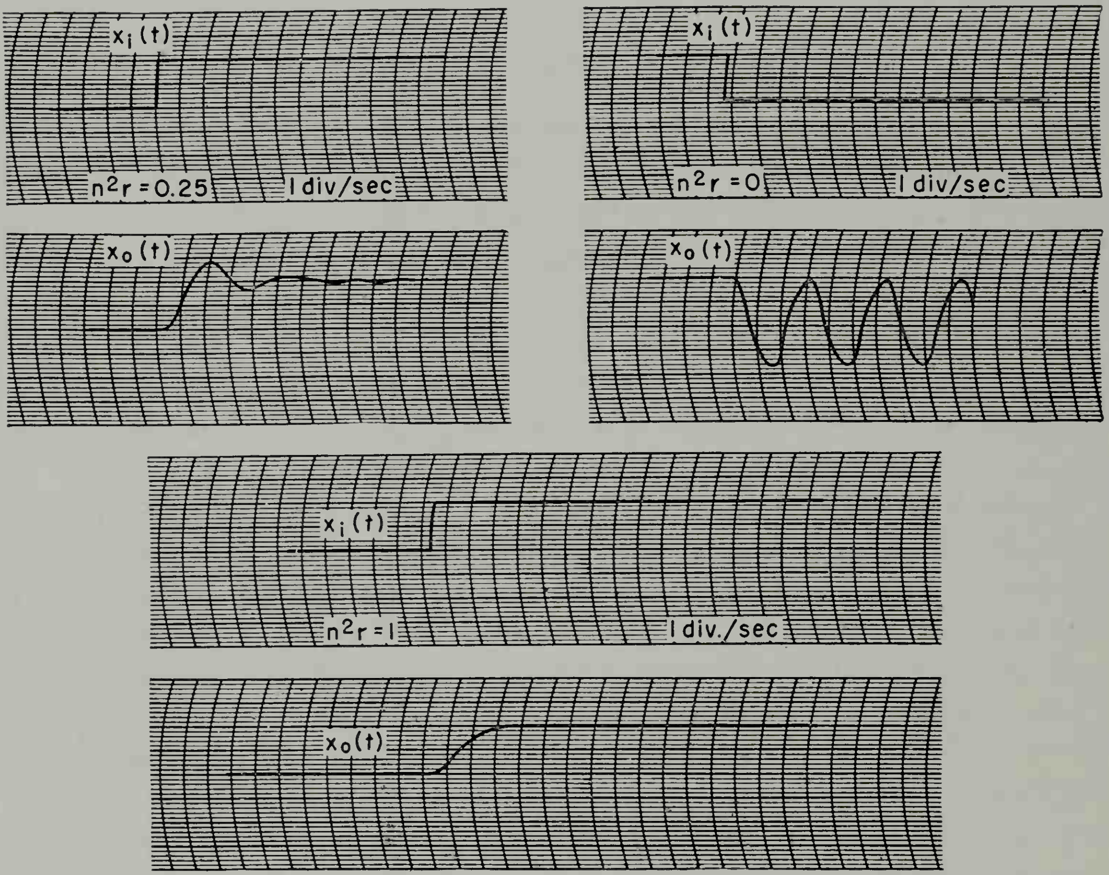

FIG. 3.4. D-c analog-computer records (Hagelbarger, Howe, and Howe, *op. cit.*), showing the response of the servomechanism of Fig. 3.3 to a step-function input displacement for three values of the motor-damping coefficient r.

The computer block diagram of Fig. 3.5*c* has been intentionally drawn so as to show the relatively close analogies between blocks of computing elements and components of the actual servomechanism. The path of

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[1] *Ibid.* James, Nichols, and Phillips, *op. cit.*

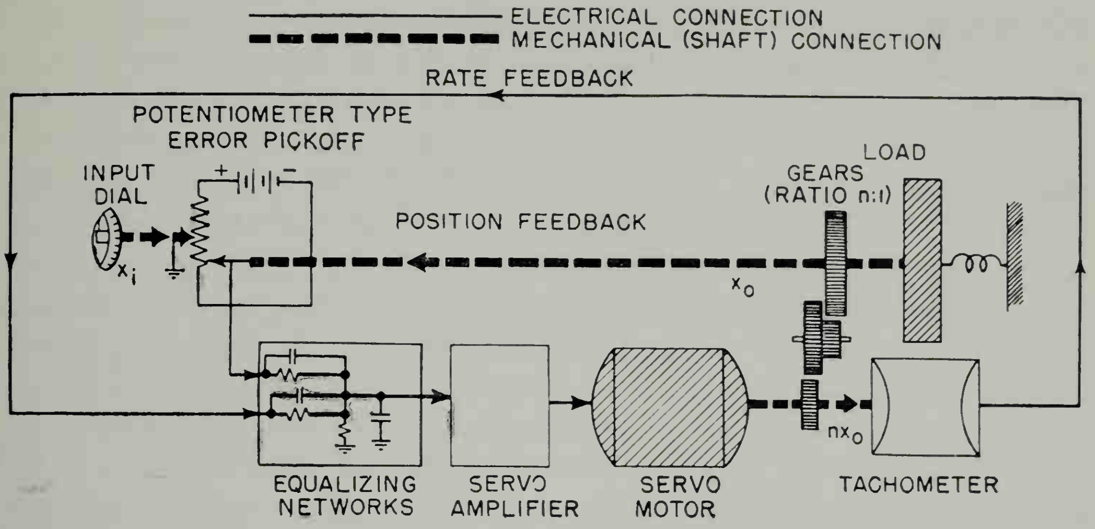

Fig. 3.5*a*. Semischematic representation of a linear servomechanism with a spring-inertia type load, rate feedback, and equalizing networks.

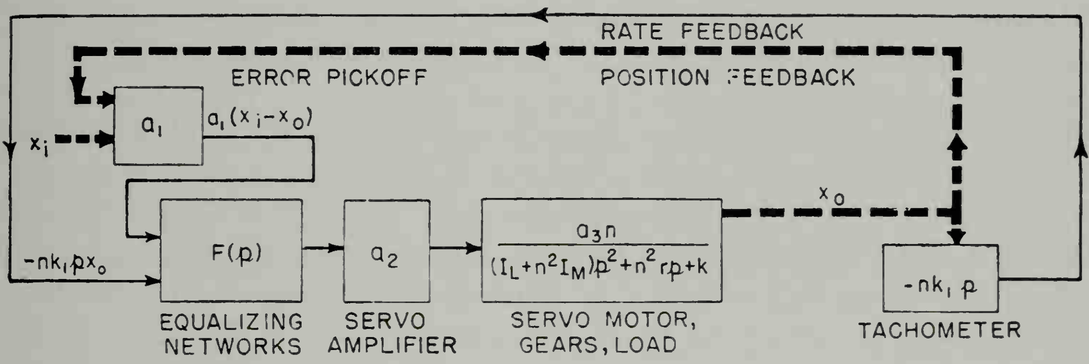

Fig. 3.5*b*. Functional block-diagram representation of the servomechanism of Fig. 3.5*a*.

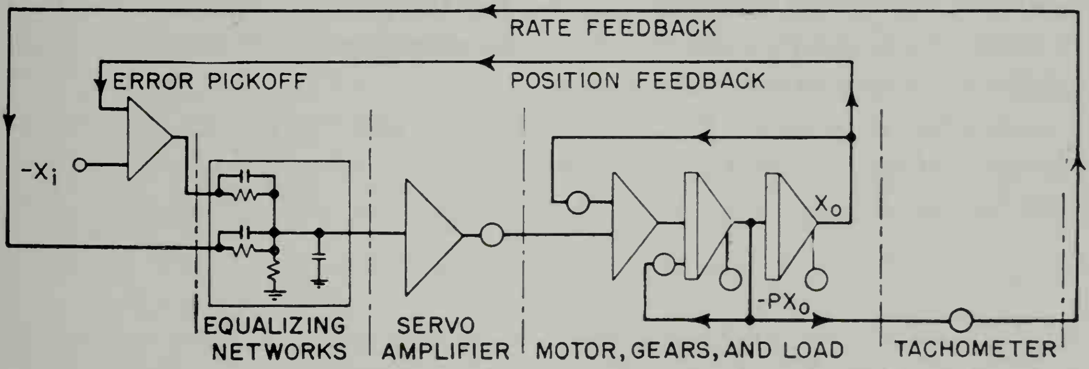

Fig. 3.5*c*. Block diagram of a d-c analog-computer setup for solving the performance equations of the servomechanism of Fig. 3.5*a*. This block diagram was drawn intentionally so as to emphasize the analogies between certain components of the actual servomechanism and corresponding blocks of computing elements.

the control signal can thus be traced through the computer setup of Fig. 3.5c just as it can be traced through the actual servomechanism.

Partial System Tests. In the computer setup of Fig. 3.5c, the performance equations of the equalizing networks have not been established by means of d-c integrators and summing amplifiers, but the *actual networks* to be used in the servomechanism have been introduced into the computer setup. The analog nature of the machine variables in analog computers makes it possible to include actual system components in computer setups. It is thus possible to perform *partial system tests* of servo amplifiers, motors, error pickoff devices, etc. Investigations of the effects of different component designs as well as tests of control-system components can thus be made under very realistic conditions (see also Sec. 3.7). D-c analog computers are well suited for such partial system tests, and this is one of their particularly desirable features.

Analysis and Synthesis of Servomechanisms: Evaluation of Computed Transient Response Records. In parameter studies of automatic control systems, such as the simple study of optimum damping illustrated by the computer records of Fig. 3.4, it becomes necessary to formulate qualitative criteria or quantitative figures of merit which will permit definite choices of "optimum" combinations of the design parameters on the basis of the corresponding computer records. Most observers will agree that the servo output displacement x_o in the third computer record of Fig. 3.4 seems to follow the input displacement x_i more faithfully than that in the second computer record; but a choice between the values of the damping constant r leading to the first and third computer records of Fig. 3.4 is more difficult. Since no one number could possibly give a complete description of the system performance, the choice of a quantitative figure of merit will depend on the nature of the particular performance characteristic most important in each case.

A number of possible figures of merit for judging the transient performance of servomechanisms are listed below. All of them are obtained more or less directly from d-c analog-computer records of the *error*

$$\epsilon(t) = x_i(t) - x_o(t)$$

obtained as a function of the time t by means of a computer setup like that shown in Fig. 3.6a, for various input signals $x_i(t)$.

1. A somewhat crude figure of merit is the reciprocal of the time taken by the absolute value of the error ϵ to "recede" below a certain specified percentage of the input voltage after a step input voltage has been applied.
2. The *displacement figure of merit* M_0 is defined as the ratio of the

output displacement x_o to the error ϵ when all derivatives of the output displacement x_o are zero. The value of M_0 is easily obtained from a computer record of $\epsilon(t)$ for a step input signal x_i (see Fig. 3.6*b*).

3. The *rate figure of merit* M_1 is defined as the ratio of the output rate px_o to the error ϵ when the output acceleration and all higher derivatives of x_o are zero. This quantity may be obtained from the computer records of $\epsilon(t)$ and px_o for a *ramp function input signal*

$$x_i = (\text{constant})\ t$$

(see Fig. 3.6*c*).

4. The *acceleration figure of merit*[1] M_2 is defined as the ratio of the acceleration p^2x_o to the error ϵ when px_o equals zero. The value of M_2 may be obtained from computer records of $\epsilon(t)$ and p^2x_o for a step input signal x_i.

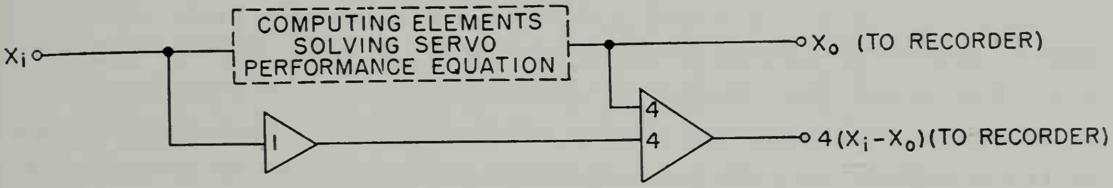

FIG. 3.6*a*. Computer setup for obtaining records of the error $\epsilon = x_i - x_o$.

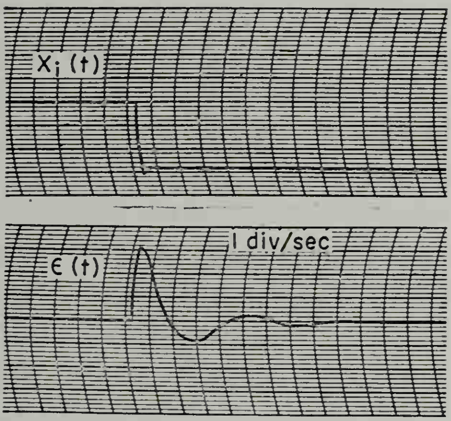

FIG. 3.6*b*. Typical error record for a step input signal (Hagelbarger, Howe, and Howe, *op. cit.*).

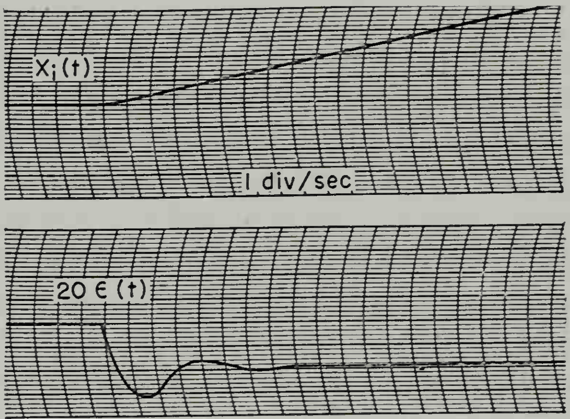

FIG. 3.6*c*. Error record for a ramp input signal (Hagelbarger, Howe, and Howe, *op. cit.*).

If the servomechanism under consideration is linear, the displacement and rate figures of merit are simply related to the first two coefficients in a power series expansion of the transfer function

$$f(p) = \frac{\epsilon}{x_i}$$

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[1] Lauer, Lesnick, and Matson, *op. cit.*

in terms of p.[1] If the servomechanism is not linear, all figures of merit may vary appreciably with the magnitude of the input signals.

A very useful measure of servo performance is given by the *average* $\overline{\epsilon^2}$ *of the square of the error* $\epsilon(t)$ over a suitable time t_1, or

$$\overline{\epsilon^2} = \frac{1}{t_1} \int_0^{t_1} \epsilon^2(t)\, dt \tag{3.9}$$

The integral (3.9) is easily computed by means of a d-c analog computer. By this criterion, the "optimum" combination of design parameters will be the one to minimize the expression (3.9) for a fixed value of the time t_1 and for a number of different input signals $x_i(t)$ considered to be typical of those encountered in actual practice.

The application of the mean-square-error criterion makes it also possible to take the effects of noise, stray input signals, and disturbing torques in the servomechanism on the choice of the "optimum" combinations of design parameters[2] into account. In parameter studies of servomechanisms it is only necessary to introduce voltages corresponding to typical noise, spurious signals, and disturbing torques into the correct points of the computer setup and to determine the mean square error (3.9) for each combination of the design parameters. Since noise is a statistical process, the computations must be repeated a number of times with different randomly chosen noise voltages in order to gain a realistic idea of the optimum combination of design parameters. The use of a repetitive computer (see Sec. 8.5) in conjunction with a suitable noise generator may be convenient in this connection. In order to obtain a more completely useful result, the entire procedure should also be repeated for as many typical input signals $x_i(t)$ as possible.

Analysis and Synthesis of Linear Servomechanisms: Transfer Function or Frequency Analysis Methods. Powerful modern methods of servomechanism analysis[3] and synthesis are based on studies of the performance of servomechanisms components for steady-state sinusoidal oscillations of their input and output signals. More specifically, the ratio of the output and input amplitudes as well as the phase difference between the sinusoidal input and output oscillations are plotted as functions of the circular frequency ω of the oscillations, and the properties of the component or group of components in question are described in terms of these *frequency-response characteristics.*

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[1] Hagelbarger, D. W., C. E. Howe, and R. M. Howe, *Investigation of the Utility of an Electronic Analog Computer in Engineering Problems*, UMM-28, Engineering Research Institute, University of Michigan, Ann Arbor, Apr. 1, 1949.

[2] James, Nichols, and Phillips, *op. cit.*

[3] Greenwood, Holdam, and MacRae, *op. cit.;* James, Nichols, and Phillips, *op. cit.*

The transient response of a linear servomechanism can be related to its frequency-response characteristics by means of Fourier analysis. It is true that the transient response of a servomechanism can be obtained directly by means of a d-c analog computer. The frequency-response method of describing servo performance is, nevertheless, particularly useful for the synthesis of linear servomechanisms because of the simple relations existing between the individual frequency-response characteristics of "transducers," such as amplifiers, networks, motors, etc., and the frequency-response characteristics of combinations of such devices connected in cascade. The amplitude ratio of the combination, for each value of the circular frequency, will be the product of the individual amplitude ratios, whereas the phase shifts are simply added. These relations help the designer to choose, for instance, the type of equalizing networks he must insert into a feedback loop in order to obtain a desired frequency response. The frequency characteristics of a servomechanism frequently not only show up undesirable conditions but also indicate what to do about them. No similarly simple relationships exist between the transient responses of individual and cascaded transducers.

D-c analog computers can be at least as useful for computing the frequency-response characteristics of servomechanism components and complete servomechanisms as they are for computing transient responses. After the performance equations of the devices in question have been set up in the usual manner, it is only necessary to provide a source of sinusoidal input signals and a means for measuring the relative amplitudes and phases of the input and output voltages. Although ordinary computer records of the input and output signals will serve this latter purpose, it is preferable to use a convenient signal generator capable of generating sinusoidal voltages of different frequencies and of measuring the amplitude and phase of an output signal by comparison with a reference signal. A signal source of this type is shown in Fig. 6.34.

An interesting application of such d-c analog-computer methods is the automatic computation of loop transfer functions[1] for use in phase-decibel plots and other powerful graphical methods of control-system analysis and synthesis.

Frequency analysis methods apply, strictly speaking, only to devices whose performance equations are linear; but the frequency-response characteristics of many nonlinear devices are at least qualitatively interesting. Sometimes a linear or nonlinear transducer whose performance equation is not known can be represented in the computer by a block of operational amplifiers or networks having similar frequency-response characteristics for most input amplitudes.

[1] James, Nichols, and Phillips, *op. cit.*

Representation of Nonlinear Transfer Characteristics. Although the important properties of many control systems can be studied in terms of linear relationships between the system variables, it is frequently absolutely necessary to explore the effects of *nonlinear* relationships existing between the input and output signals of certain control-system components. The use of analog computers for such investigations is particularly important because it may be impossible to find valid analytical formulations for the relationships existing between the problem variables, whereas many such relationships can be established by means of relatively simple function generators in an analog computer. In studies of automatic control systems, it is often sufficient to represent the nonlinear transfer characteristics of control system components to an accuracy between 1 and 5 per cent. Under these conditions, function generators based on the properties of simple diode switches are particularly versatile and useful, since they do not contain moving parts and have good high-frequency-response characteristics. The properties of such function generators are discussed in detail in Sec. 6.7; several particularly interesting applications of diode-type function generators to the simulation of control-system components are presented below. If the problem in question is run on a very slow time scale, it is also possible to use *computing relays* (see Sec. 6.5) instead of diode switches in many of the applications.

Position limits (stops), rate limits, and torque limits in servomechanisms may be represented in d-c analog-computer setups by means of limiting diodes in the manner discussed in Sec. 3.2. As an example, consider the computer setup for analyzing the performance of a hydraulic controller. The controller, which may replace the electric motor in the servomechanisms shown in Figs. 3.3 and 3.5, consists of a solenoid valve whose displacement y is controlled by an input voltage e, and a hydraulic actuator whose piston displacement x_o is, in turn, controlled by the valve displacement. The performance equations of the hydraulic controller may frequently be approximated in the form[1]

$$x_o = \frac{a_2}{p} y \qquad y = \frac{a_1}{(bp + 1)(mp^2 + rp + k)} e \tag{3.10}$$

where a_1 and a_2 are gain coefficients, b is the time delay due to the solenoid, and m, r, and k are the mass, damping coefficient, and spring constant, respectively, of the valve. The rate px_o of the output motion is limited by the flow capacity of the hydraulic system, and the piston displacement

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[1] McCann, G. D., C. H. Wilts, and B. N. Locanthi, *Trans. AIEE*, **68**, Paper 49–165, 1949. For a more complete discussion of hydraulic servomechanisms, see M. R. Hannah, Frequency Response Measurements of a Hydraulic Power Unit, Paper 47-A-70, ASME Annual Meeting, Dec. 1–5, 1947.

x_o is limited by mechanical stops. The form of a d-c analog-computer setup corresponding to Eq. (3.10) is indicated in Fig. 3.7.

Diode limiters or more complex function generators based on diode characteristics (see Sec. 6.7) may also be used for simulating the *saturation characteristics* of overdriven servo amplifiers and motors.

Figure 3.8*a* illustrates the use of a high-gain d-c amplifier and a pair of limiting diodes for simulating the generation of a *dry friction torque* as a function of speed. The functional relationship between the input voltage X_1 (proportional to speed) and the output voltage X_2 (proportional to torque) is indicated in the graph on the right of Fig. 3.8*a*.

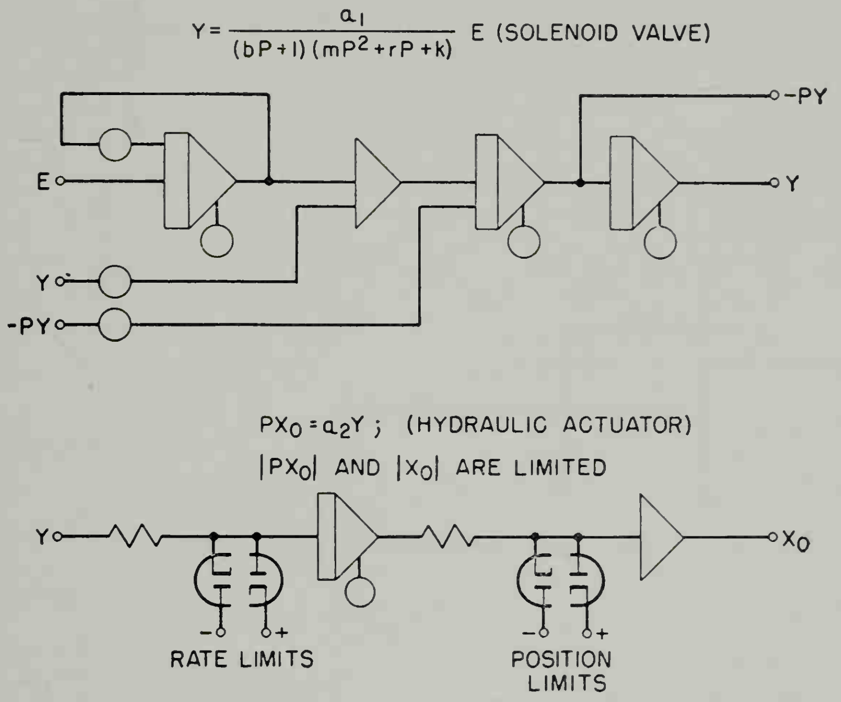
Y = a₁ / (bP+1)(mP²+rP+k) E (SOLENOID VALVE)
-PY
E
Y
Y
-PY
PX₀ = a₂Y ; (HYDRAULIC ACTUATOR)
|PX₀| AND |X₀| ARE LIMITED
Y
X₀
- +
RATE LIMITS
- +
POSITION LIMITS

FIG. 3.7. Block diagram of a d-c analog-computer setup approximately simulating a hydraulic controller with rate and position limiting.

Figure 3.8*b* shows a computer setup for the simulation of an *inert-zone characteristic.* Inert-zone characteristics are encountered in many mechanical or electromechanical transducers, such as tachometers, gyros, and pressure gauges. It is seen that the input signal must reach a certain positive or negative *threshold level* before the output signal can follow.

In the computer setup of Fig. 3.8*c*, the transfer characteristics of Fig. 3.8*a* and *b* have been combined with a "time-delay circuit" having the transfer function

$$\frac{X_2}{X_1} = -\frac{k}{bP + 1}$$

where k and b are constants. The resulting configuration approximates the relation between the input and output voltages of a *control relay circuit* or between the input voltage and output shaft speed of a *solenoid-operated dry friction clutch.*

Many other types of nonlinear characteristics may be approximated by the performance of circuits containing diode switches (see Sec. 6.7). As a particularly interesting example, Fig. 3.9 shows two circuits capable of simulating the relation between the displacements ϑ_1 and ϑ_2 of two *gears with backlash.*

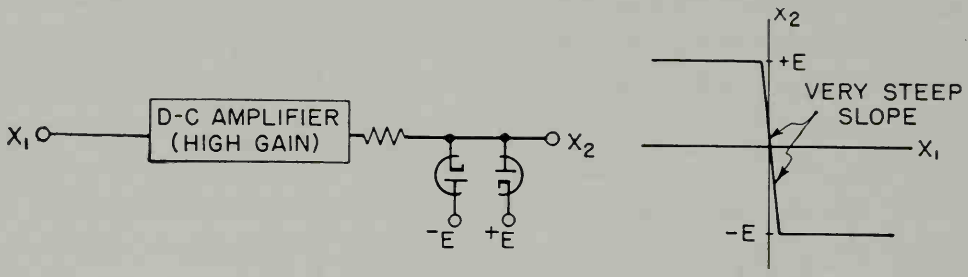

FIG. 3.8*a*. Simulation of dry friction torque.

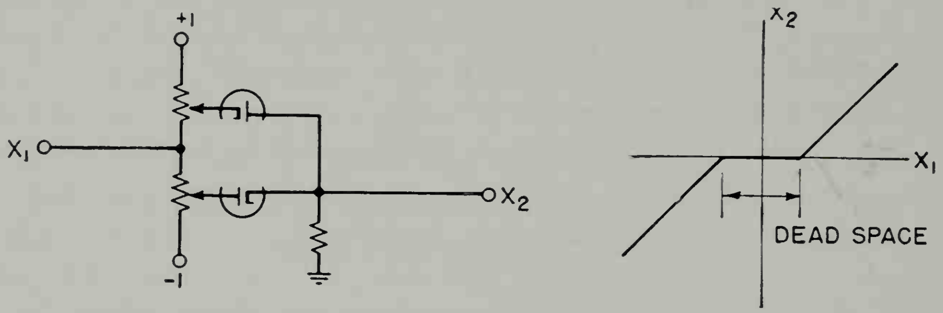

FIG. 3.8*b*. Simulation of dead space.

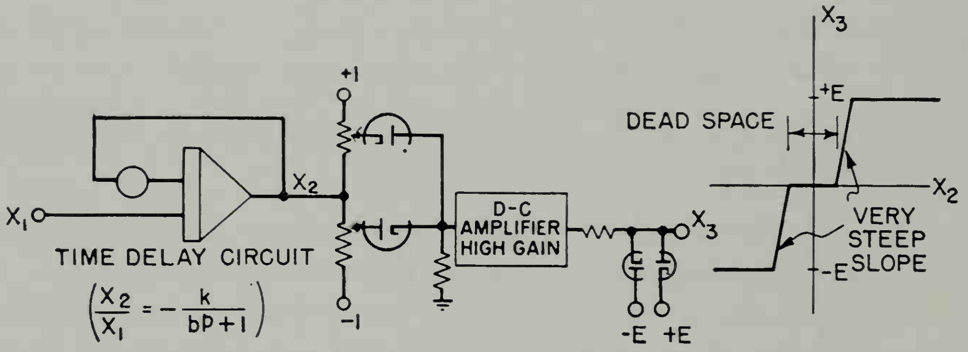

FIG. 3.8*c*. Simulation of a relay or of a solenoid clutch.

The input and output displacements ϑ_1 and ϑ_2 of such a gear train are represented by the voltages Θ_1 and Θ_2, respectively, and half the backlash "gap" is represented by a voltage E. Referring to Fig. 3.9*a*, it is seen that the lower diode will conduct, and

$$\Theta_2 = \Theta_1 - E \qquad \text{if and only if } \Theta_1 - \Theta_2 > E$$

Again, the upper diode will conduct, and

$$\theta_2 = \theta_1 + E \qquad \text{if and only if } \theta_1 - \theta_2 < -E$$

As long as either diode conducts, the time constant of the circuit is so short that the presence of the capacitor C will cause no appreciable integration.

If, however,

$$-E \leq \theta_1 - \theta_2 \leq E$$

neither diode will conduct, and the capacitor C will tend to store or "remember" the voltage θ_2 previously imparted to it unless leakage is

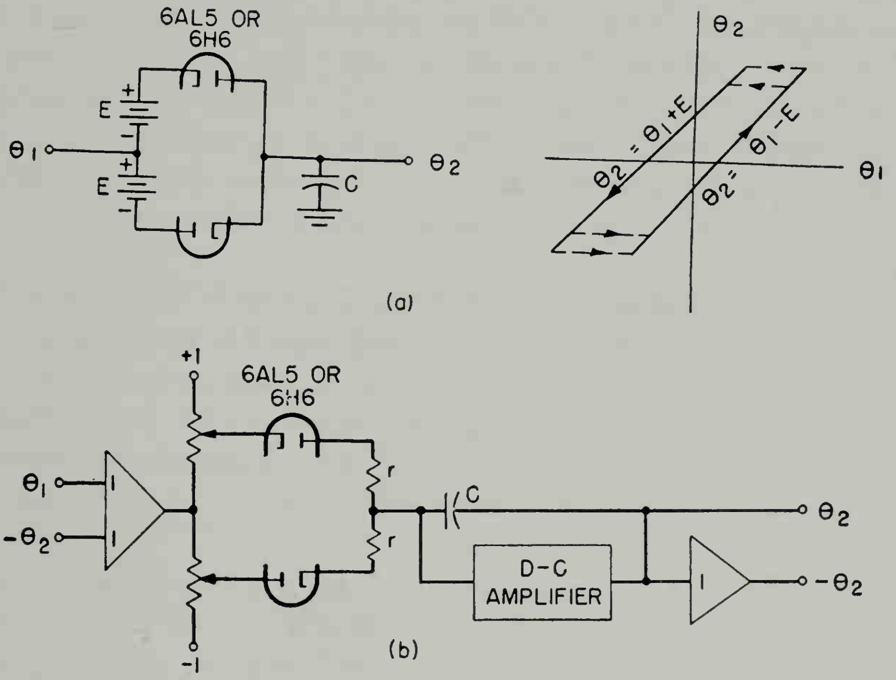

FIG. 3.9. Simulation of a gear train with backlash through the use of series diode circuits. Combinations of such circuits with suitable function generators permit simulation of hysteresis loops, thermodynamic cycles, etc.

excessive. The resultant transfer characteristics are shown on the right of Fig. 3.9*a* and closely approximate those of a gear train with backlash.

The circuit of Fig. 3.9*b* is a more practical version of the basic circuit of Fig. 3.9*a*. The storage capacitor C is arranged so that it cannot be discharged through the load connected to the output terminal, and the diode bias voltages are obtained from the computer reference power supplies instead of from battery sources. The resistors r shown are current-limiting resistors of a few hundred ohms.

The computer setups of Fig. 3.9 are examples of transducers whose transfer characteristics depend on the "direction" of the input rate of change. Many other similar arrangements are possible. The addition of simple output limiters to the computer setups of Fig. 3.9 permits the simulation of a *hysteresis loop with saturation* such as those typical of magnetic circuits. Similar arrangements make it possible to simulate the performance cycles of heat engines, pumps, storage batteries, and other similar processes.

Accuracy Requirements. In many investigations of servomechanisms by means of d-c analog computers, it is sufficient to obtain *qualitative* results with respect to the stability and relative errors of different system configurations. For this reason, a great deal of useful information has been obtained through the use of very cheap computing elements. McDonald[1] has pointed out that, in most d-c analog-computer analyses of servomechanisms, the gain changes and phase shifts resulting from the use of low-cost operational amplifiers (see Chap. 4) usually enter the computation in such a manner that they are negligible compared with the gain changes and phase shifts inherent in the transfer functions of the simulated servomechanisms. This question should, however, be investigated in detail in each particular case if valid *quantitative* results are desired.

3.4. D-c Analog-computer Setup for a Trajectory Computation: The Trajectory of a Slow Spherical Cannonball with Air Resistance. *Introduction.* As a simple example of a *trajectory computation* in two dimensions, consider the computation of the trajectory of an old-fashioned spherical cannonball with air resistance. This example will serve as an introduction to more complex ballistic computations involving more complicated projectiles such as rockets.

The cannonball is fired from a cannon with a given muzzle velocity v_0 and at a given elevation angle ϑ_0. It is desired to find the trajectory of the cannonball with respect to a rectangular Cartesian coordinate system whose x and y axes are horizontal and vertical, respectively, and whose origin is located at the cannon muzzle (trajectory origin).

The only forces acting on the cannonball are the force of gravity and the aerodynamic drag. The latter force is assumed to be proportional to the square of the instantaneous velocity of the projectile. Since the aerodynamic force acting on a body in flight is usually specified in terms of its components in the direction of the velocity vector (drag) and at right angles to this direction (lift and side force), it is often convenient to write the equations of motion of the projectile with respect to "wind axes" along and perpendicular to the velocity vector. Let ϑ be the angle between the horizontal and the positive instantaneous velocity direction. Referring to Fig. 3.10*a* and to the glossary of symbols (see page 78) the

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[1] McDonald, D., Analog Computers for Servo Problems, *Rev. Sci. Instruments*, **21**, February, 1950.

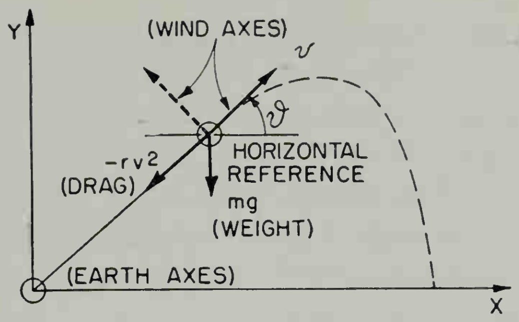

FIG. 3.10*a*. Computation of the trajectory of a slow spherical cannonball.

$$10PV = -0.161 \sin\left(\frac{\pi}{2}\Theta\right) - 0.378V^2$$

$$10P\Theta = -1.03\left(\frac{1}{10V}\right)\cos\left(\frac{\pi}{2}\Theta\right)$$

$$10PX = 0.238V\cos\left(\frac{\pi}{2}\Theta\right)$$

$$10PY = 0.834V\sin\left(\frac{\pi}{2}\Theta\right)$$

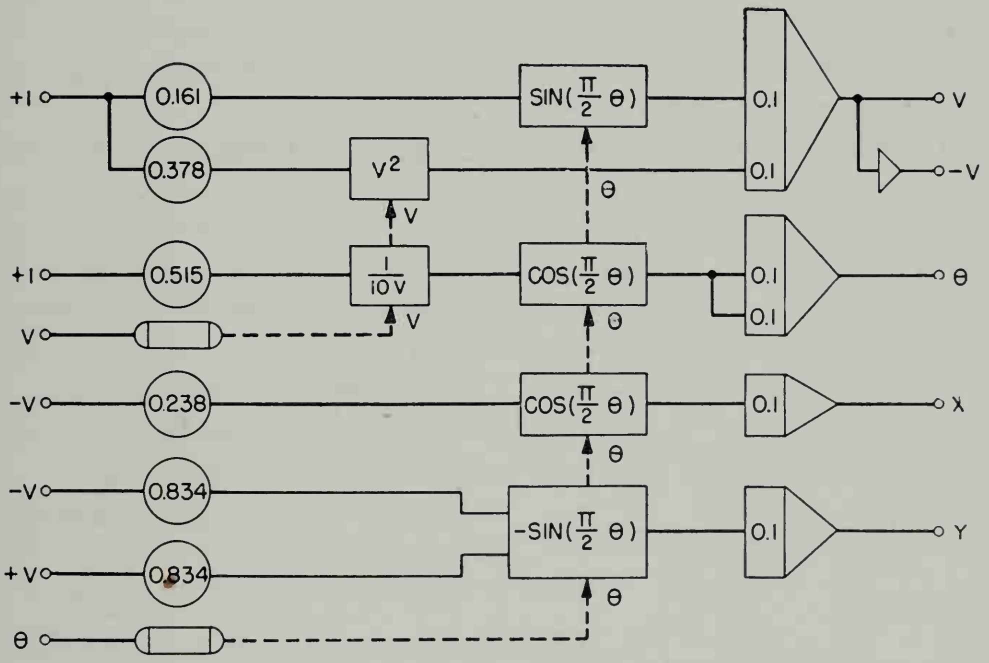

FIG. 3.10*b*. Block diagram of a possible computer setup for the trajectory computation. Potentiometer-loading corrections must be used if good accuracy is desired. Solid lines indicate electrical connections, and broken lines indicate mechanical connections (servo shafts). No initial condition settings are indicated.

equation of motion in the positive velocity direction is

$$m\frac{dv}{dt} = -mg\sin\vartheta - rv^2$$

The equation of motion (upward) at right angles to the velocity direction may be expressed in the form

$$mv\frac{d\vartheta}{dt} = -mg\cos\vartheta$$

Note that the acceleration in the direction perpendicular to the instantaneous velocity is equal to $v(d\vartheta/dt)$.

These two differential equations permit the computation of the projectile velocity v and the angle ϑ as functions of the time t. The instantaneous Cartesian coordinates x and y of the cannonball are then obtained from v and ϑ through the relations

$$\frac{dx}{dt} = v\cos\vartheta \qquad \frac{dy}{dt} = v\sin\vartheta \tag{3.11}$$

A practical procedure for handling this trajectory problem in an analog-computer laboratory is illustrated below in terms of a sample *problem statement* and a set of sample *computer setup records.*

Trajectory Computation for a Slow Spherical Cannonball

A. Problem specifications

1. *Introduction.* This paragraph will contain problem designations, authorization references, etc.
2. *Statement of the problem.* It is desired to obtain the parametric representation

$$x = x(t) \qquad y = y(t)$$

of the trajectory of a slow spherical cannonball whose equations of motion are given below.

3. *Mathematical Relations*

$$(a)\quad m\frac{dv}{dt} = -mg\sin\vartheta - rv^2$$

(Motion in the direction of the instantaneous velocity)

$$(b)\quad mv\frac{d\vartheta}{dt} = -mg\cos\vartheta$$

(Motion at right angles to the instantaneous velocity)

$$\left.\begin{aligned}(c)\quad \frac{dx}{dt} &= v\cos\vartheta\\ (d)\quad \frac{dy}{dt} &= v\sin\vartheta\end{aligned}\right\}$$

(Transformation to earth axes)

4. *Glossary of symbols*

m	mass of cannonball
t	time
$v(t)$	velocity of cannonball
$-rv^2$	aerodynamic drag
g	acceleration of gravity
$\vartheta(t)$	angle between velocity vector and horizontal

$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ rectangular Cartesian coordinates of cannonball in the horizontal and vertical directions, respectively

v_0 value of $v(t)$ for $t = 0$

ϑ_0 value of $\vartheta(t)$ for $t = 0$

5. *Initial conditions.* For $t = 0$, $x = y = 0$

$$\left.\frac{dx}{dt}\right]_0 = v_0 \cos \vartheta_0$$

$$\left.\frac{dy}{dt}\right]_0 = v_0 \sin \vartheta_0$$

See below for the values of v_0 and ϑ_0 to be used.

6. *Numerical values of constants*

$g = 32.2$ ft/sec^2

$m = 0.621$ slug (20 lb)

$r = 4.7 \times 10^{-5} \frac{\text{lb}}{\text{ft}^2} \text{sec}^2$

7. *Estimated ranges of variables*

$0 < t < 25$ sec

$0 < x < 21{,}000$ ft

$0 < y < 6{,}000$ ft (negative values of y are not of interest)

$100 < v < 1{,}000$ ft/sec

$-45° < \vartheta \leq 21°$

8. *Parameters to be varied*

Take $v_0 = 1{,}000$, 950, and 1,050 ft/sec

Take $\vartheta_0 = 20°$, 19°, and 21°

(Nine solutions required)

9. *Expected results of parameter variations.* Increases in the value of v_0 are expected to increase the values of x and y for each value of the time t.

10. *Possible analytical or numerical checks noted by the customer.* If there is no air resistance ($r = 0$), then

$$x = v_0 t \cos \vartheta_0 \qquad y = v_0 t \sin \vartheta_0 - \tfrac{1}{2} g t^2$$

11. *Use of results.* To prepare ballistic tables for an eighteenth-century cannon. In particular, the sensitivity of the trajectory to small variations in the initial conditions is of interest.

12. *Accuracy required.* Better than 3 per cent.

B. *Record of computer solution*

1. *Transformation equations.* (x, y, v, and t are measured in the ft/slug/sec system; ϑ is measured in radians; all machine variables are measured in machine units. One machine unit equals 100

volts, or a servo shaft rotation corresponding to 100 volts input to the servomechanism.)

$$x = 21{,}000X \qquad y = 6{,}000Y$$

$$v = 1{,}000V \qquad \vartheta = \frac{\pi}{2}\Theta$$

$$t = \tfrac{1}{2}\tau \qquad \frac{d}{dt} = 2P$$

NOTE: The choice of the scale factor for ϑ depends on the types of function potentiometers used to generate voltages proportional to $\sin \vartheta$ and $\cos \vartheta$. The choice made here permits the use of commercial sine and cosine potentiometers whose wiper positions correspond to $\vartheta = \pm \pi/2$ when the input voltage Θ to the servomechanism equals plus or minus 1 machine unit, respectively.

The 2:1 (slow) time scale was chosen tentatively to permit greater accuracy in the computer servomechanisms.

The recorder scales are noted on the records.

2. *Machine equations*

$$(a)\quad 10PV = -0.161 \sin\left(\frac{\pi}{2}\Theta\right) - 0.378V^2$$

$$(b)\quad 10P\Theta = -0.103 \frac{1}{V} \cos\left(\frac{\pi}{2}\Theta\right)$$

Introducing $1/V = 10(1/10V)$ to permit the use of a function potentiometer (see Sec. 2.2), (*b*) becomes

$$(b)\quad 10P\Theta = -1.03 \left(\frac{1}{10V}\right) \cos\left(\frac{\pi}{2}\Theta\right)$$

$$(c)\quad 10PX = 0.238V \cos\left(\frac{\pi}{2}\Theta\right)$$

$$(d)\quad 10PY = 0.834V \sin\left(\frac{\pi}{2}\Theta\right)$$

3. *Machine initial conditions*

$$X_0 = 0 \qquad Y_0 = 0 \qquad V_0 = 0.001v_0 \qquad \Theta_0 = 0.637\vartheta_0$$

4. *Block diagram* (see Fig. 3.10*b*). Integrators employing 10 μf integrating capacitors are indicated to facilitate operation on a reduced time scale. Loading corrections for potentiometers should be observed.
5. *Discussion* (of special problems encountered during the computation, tentative conclusions from computer results, etc.)
   None.

6. *Enclosure.* Dated and initialed computer records (x vs. t, y vs. t, and y vs. x).

More General Trajectory Computations. The foregoing example will serve to illustrate a method for solving ballistic trajectory problems on d-c analog computers. If the simple cannonball is replaced by a rocket or other self-propelled body, the effects of mass changes and thrust forces will have to be considered.[1] In the case of more complicated nonspherical projectiles, the effects of the aerodynamic lift as well as of various control forces must be taken into consideration. In addition, the "moment equation" describing the rotation of the projectile about the pitch axis must be solved, and the interaction between this rotation and the translatory motion must be determined. An elementary discussion of some of these questions for a special case is presented in Sec. 3.5. Many three-dimensional as well as two-dimensional trajectory problems can be solved successfully by means of d-c analog computers subject to the accuracy limitations of the latter.

3.5. Introduction to D-c Analog-computer Methods for Solving Aircraft Flight Equations.

Description of the Motion of an Aircraft: Motion in the Pitch Plane. The motion of an aircraft in the atmosphere, like the motion of the projectile discussed in Sec. 3.4, may be conveniently specified by reference to *wind axes* along and perpendicular to the direction of the instantaneous velocity vector and centered in the center of gravity of the aircraft; the velocity direction is, of course, opposite to that of the relative wind in a stationary air mass.

As an important special case, it will be interesting to consider the motion of an aircraft which is not turning, slipping, or skidding. The motion will, then, take place in the vertical plane, pictured in Fig. 3.11. The *trajectory* of the aircraft center of gravity in this "pitch plane" may be described, in the manner shown in Sec. 3.4, by specifying the magnitude $v(t)$ of the instantaneous velocity and the angle $\vartheta(t)$ between the velocity direction and the horizontal as functions of the time t. The rectangular Cartesian coordinates $x(t)$ and $y(t)$ of the trajectory are, again, related to $v(t)$ and $\vartheta(t)$ by Eqs. (3.11), or

$$\frac{dx}{dt} = v \cos \vartheta \qquad \frac{dy}{dt} = v \sin \vartheta \tag{3.12}$$

In addition to the motion of the aircraft center of gravity, the pitch attitude angle $\varphi(t)$, defined as the angle between some preferred longitudinal axis fixed in the aircraft and the horizontal reference, must be specified as a function of the time t.

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[1] See J. B. Rosser, R. R. Newton, and G. L. Gross, *Mathematical Theory of Rocket Flight*, McGraw-Hill, New York, 1947, for a detailed discussion of the equations of motion of rockets.

The angle $\gamma(t)$ between the longitudinal axis and the velocity direction, or

$$\gamma(t) = \varphi(t) - \vartheta(t) \tag{3.13}$$

is known as the pitch *angle of attack.*

The motion of the aircraft in the pitch plane can be *controlled* by changing the deflection angles of control surfaces such as elevators, spoilers, elevons, or movable wings and by changing the magnitude, direction, or point of attack of the thrust propelling the aircraft. The various control parameters will be denoted by δ_1, δ_2, . . . , etc.

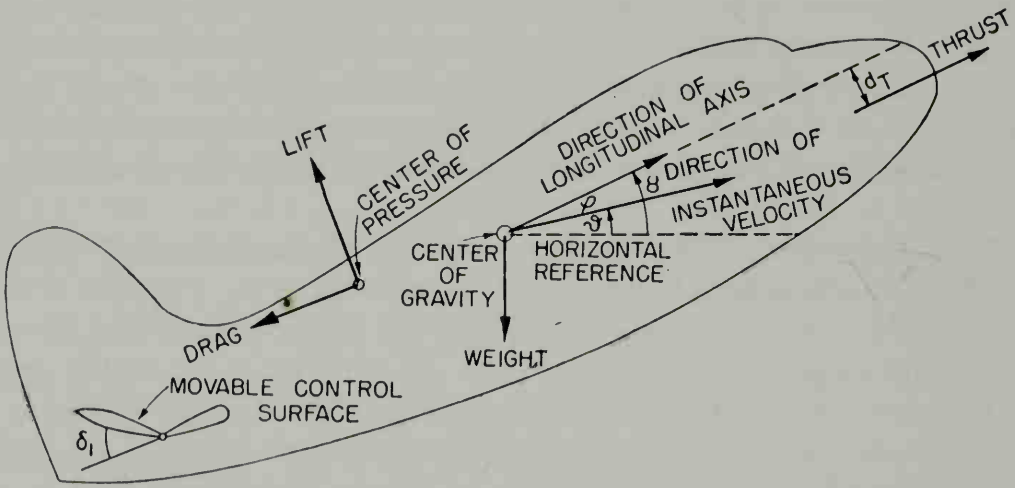

FIG. 3.11. Description of the two-dimensional motion of an aircraft in the pitch plane. Note that neither the location of the center of pressure nor of the center of gravity is necessarily fixed with respect to the airframe.

Flight Equations for Motion in the Pitch Plane. The system of forces acting on the aircraft can usually be reduced to the four forces indicated in Fig. 3.11. These may be listed as follows:

1. The *thrust* T due to the power plant or power plants. In most conventional aircraft, the thrust direction and point of attack are not used for control purposes. The longitudinal axis may then be chosen to coincide with the thrust axis. The latter does not necessarily pass through the center of gravity of the aircraft.
2. The *weight* mg of the aircraft, where m is the aircraft mass and g is the acceleration of gravity.

3. The *drag* D is the component of the total aerodynamic force in the negative velocity direction. For each particular aircraft configuration, the drag will appear as a function

$$D = D(v,\gamma,\rho) \tag{3.14}$$

of the velocity v, the angle of attack γ, and the air density ρ, and possibly also of some of the control parameters δ_1, δ_2, ...

4. The *lift* L is the component of the total aerodynamic force at plus 90 deg with respect to the velocity direction. For each particular aircraft configuration, the lift will appear as a function

$$L = L\left(v,\gamma, \frac{d\varphi}{dt}, \delta_1,\delta_2, \ . \ . \ . \ ,\rho\right) \tag{3.15}$$

of the velocity v, the angle of attack γ, the pitching rate $d\varphi/dt$, the control surface deflections δ_1, δ_2, . . . , and the air density ρ.

In the case of certain jet aircraft, and especially of rockets, the aircraft cannot be regarded as a rigid body but contains a large amount of moving fuel and exhaust gases which may subject it to various kinetic reaction forces in addition to the jet thrust;[1] these forces are not considered in the analysis presented below.

The equation of motion of the aircraft in the positive velocity direction is, by reference to Fig. 3.11,

$$m\frac{dv}{dt} = T\cos\gamma - mg\sin\vartheta - D(v,\gamma,\rho) \tag{3.16}$$

The equation of motion at right angles to the instantaneous velocity direction may be written in the form

$$mv\frac{d\vartheta}{dt} = -mg\cos\vartheta + L\left(v,\gamma, \frac{d\varphi}{dt}, \delta_1,\delta_2, \ . \ . \ . \ ,\rho\right) \tag{3.17}$$

The equation of motion describing the changes of the pitch angle φ takes the form

$$I\frac{d^2\varphi}{dt^2} = d_T T + M\left(v,\gamma, \frac{d\varphi}{dt}, \delta_1,\delta_2, \ . \ . \ . \ ,\rho\right) \tag{3.18}$$

where M is the *pitch moment* due to the aerodynamic forces and d_T is the distance between the thrust axis and the center of gravity. The moment

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[1] See, for instance, *ibid.*

M may be computed from known values of the drag D and the lift L if the location of the point of attack of the total aerodynamic force (center of pressure, see Fig. 3.11) is known, but it is usually simpler to obtain M as a function of v, γ, $d\varphi/dt$, δ_1, δ_2, . . . , and ρ directly from wind-tunnel data.

The three flight equations (3.16), (3.17), and (3.18), in conjunction with the definition (3.13) of the angle of attack γ, determine the motion of the aircraft in the pitch plane and permit one to compute $v(t)$, $\vartheta(t)$, $\varphi(t)$, and $\gamma(t)$ as functions of the time t if proper initial conditions are given.

D-c Analog-computer Solution of the Flight Equations. The flight equations are usually suitable for d-c analog-computer solution. The equations of motion of the projectile discussed in Sec. 3.4 are special cases of the flight equations (3.16) and (3.17); the d-c analog-computer setup discussed in Sec. 3.4 is thus an example of the d-c analog-computer solution of a flight problem.

At least for low values of the velocity v, it is often possible to express the functions D, L, and M to a good approximation in the form

$$\left.\begin{aligned} D(v,\gamma,\rho) &= c_1\rho D_1(v)D_2(\gamma) \\ L\left(v,\gamma,\frac{d\varphi}{dt},\delta_1,\delta_2,\ \ldots\ ,\rho\right) &= c_2\rho L_1(v)\left[L_2(\gamma) + L_3\left(\frac{d\varphi}{dt}\right)\right. \\ &\qquad\left. + L_4(\delta_1) + \cdots\right] \\ M\left(v,\gamma,\frac{d\varphi}{dt},\delta_1,\delta_2,\ \ldots\ ,\rho\right) &= c_3\rho M_1(v)\left[M_2(\gamma) + M_3\left(\frac{d\varphi}{dt}\right)\right. \\ &\qquad\left. + M_4(\delta_1) + \cdots\right] \end{aligned}\right\} \tag{3.19}$$

where c_1, c_2, and c_3 are constants. The expressions (3.19) involve only functions of single variables and are particularly easily computed by means of servo-driven function potentiometers which are capable of simultaneous multiplication and function generation. D-c analog-computer solutions of the flight equations based on these techniques can be successfully applied to the computation of aircraft trajectories even in the case of more or less complicated maneuvers such as stalls, landings, and rocket-assisted take-offs.

Aircraft Motion in Three Dimensions. If the aircraft motion under consideration is not confined to the pitch plane, a total of six generalized coordinates must be specified as functions of the time through a set of six flight equations.[1] The solution of the general problem by means of a d-c analog computer is not more difficult in principle than the determination of the motion in the pitch plane, but the number of computing elements needed in the former case will be materially larger. The transformations relating the variation of the three-dimensional velocity vector to the Cartesian coordinates of the three-dimensional trajectory will also involve more complex trigonometric expressions than the corresponding transformation equations (3.12) for motion in the pitch plane.

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[1] See, for instance, R. von Mises, *Theory of Flight*, McGraw-Hill, New York, 1945.

Flight Simulators. An analog computer capable of solving aircraft flight equations on a 1:1 time scale is frequently referred to as a *flight simulator.* If input voltages corresponding to the control surface deflections are supplied to the machine either manually or by means of appropriate control mechanisms, the machine variables describing the attitude and trajectory of the aircraft must vary as the corresponding aircraft coordinates would in actual flight. These voltages may be recorded for later study of the simulated aircraft performance and may also actuate real or simulated flight instruments.

The operation of a flight simulator on the ground is vastly cheaper as well as safer than operation of actual aircraft. For this reason, analog computers are used in increasing numbers as flight simulators for the following purposes:

1. *Training.* Flight simulators are invaluable as safe, rapid, and reliable synthetic training devices. The use of flight simulators will, for instance, permit realistic instruction in instrument flying, aerial gunnery, and power-plant control for single- and multiengine aircraft. Instruction can be given under controlled conditions which can be easily evaluated and repeated by the instructor.

Such synthetic training devices are also useful for evaluating flight procedures, crew cooperation, and job assignments among air crews.

2. *Aircraft design and redesign.* A flight simulator is a flexible computer which can simulate aircraft parameter changes in hundreds of "test flights" under controlled laboratory conditions.

Such flight simulation permits one to test the behavior of new aircraft configurations in standard maneuvers as well as in potentially dangerous stalls, spins, etc. If desired, the maneuvers in question can also be performed on a slow time scale.

Again, the cost of testing a modern multiengine aircraft can amount to thousands of dollars per hour, so that very substantial savings can be realized by performing as many preliminary tests as possible on a flight simulator.

3. *Partial system testing.* The output voltages of a flight simulator may be used to change the attitude of a servo-driven *flight table* or to control the "altitude" in a pressure chamber. Flight instruments such as gyros and altimeters may thus be tested under conditions closely simu-

lating those of actual flight. Flight simulators are particularly useful for the design of automatic pilots (see Sec. 3.7).

Because of the limitations imposed on the computing time by d-c analog computing elements (see Chap. 4), the latter may not be so suitable for use in flight simulators intended for training purposes as electromechanical analog computers. The relatively high accuracy, low cost, and flexibility of d-c analog computers make them, however, eminently useful as flight simulators used for design and partial system testing.

3.6. Analysis of Small Aircraft Motions in the Pitch Plane: D-c Analog-computer Solution of the Linearized Flight Equations. *Introduction.* The solution of the set of flight equations (3.16), (3.17), (3.18), and (3.13) for aircraft motion in the pitch plane can be simplified very considerably if it is necessary to determine only *small changes* $\bar{v}$, $\bar{\vartheta}$, and $\bar{\varphi}$ of the variables v, ϑ, and φ from their constant values for some condition of steady-state or equilibrium flight. Steady-state or equilibrium motion in the pitch plane corresponds to a condition of steady horizontal flight, climb, or dive given by a set of values of the three variables v, ϑ, and φ; it is desired to investigate the results of disturbing aerodynamic forces and/or control surface deflections.

Linearized Flight Equations for Motion in the Pitch Plane. Steady-state or equilibrium motion in the pitch plane may be defined by the relations

$$\frac{dv}{dt} = \frac{d\vartheta}{dt} = \frac{d\varphi}{dt} = 0 \tag{3.20}$$

It will be assumed that the elevator deflection δ_1 is the only control parameter and that the air density ρ is constant. Let $\bar{v}$, $\bar{\vartheta}$, $\bar{\varphi}$, $\bar{\gamma}$, and $\bar{\delta}_1$ be the variations of v, ϑ, φ, γ, and δ_1 from their respective equilibrium values v_1, ϑ_1, φ_1, γ_1, and δ_{11}, so that

$$\left.\begin{array}{c} v = v_1 + \bar{v} \qquad \vartheta = \vartheta_1 + \bar{\vartheta} \qquad \varphi = \varphi_1 + \bar{\varphi} \\ \gamma = \gamma_1 + \bar{\gamma} \qquad \delta_1 = \delta_{11} + \bar{\delta}_1 \end{array}\right\} \tag{3.21}$$

Since $\bar{v}$, $\bar{\vartheta}$, $\bar{\varphi}$, and $\bar{\gamma}$ are assumed to be small, one may write

$$\left.\begin{array}{l} \sin\vartheta = \sin\vartheta_1 + (\cos\vartheta_1)\bar{\vartheta} \qquad \cos\vartheta = \cos\vartheta_1 - (\sin\vartheta_1)\bar{\vartheta} \\ D(v,\gamma,\rho) = D(v_1,\gamma_1,\rho) + D_v\bar{v} + D_\gamma\bar{\gamma} \\ L\left(v,\gamma,\delta_1,\dfrac{d\varphi}{dt},\rho\right) = L(v_1,\gamma_1,\delta_{11},0,\rho) + L_v\bar{v} + L_\gamma\bar{\gamma} + L_{\delta_1}\bar{\delta}_1 + L_{\dot{\varphi}}\dfrac{d\bar{\varphi}}{dt} \\ M\left(v,\gamma,\delta_1,\dfrac{d\varphi}{dt},\rho\right) = M(v_1,\gamma_1,\delta_{11},0,\rho) + M_v\bar{v} + M_\gamma\bar{\gamma} + M_{\delta_1}\bar{\delta}_1 + M_{\dot{\varphi}}\dfrac{d\bar{\varphi}}{dt} \end{array}\right\} \tag{3.22}$$

where the derivatives of D, L, and M with respect to v, γ, δ_1, and $d\varphi/dt$ are all computed for the equilibrium values of v, γ, δ_1, and $d\varphi/dt$.

By introducing the relations (3.20), (3.21), and (3.22) into the flight equations (3.16), (3.17), (3.18), and (3.13) it may be shown that the variations $\bar{v}$, $\bar{\vartheta}$, $\bar{\varphi}$, $\bar{\gamma}$, and $\bar{\delta}_1$ must satisfy the *linearized flight equations*

$$\left.\begin{aligned}
&(a) \quad m\frac{d\bar{v}}{dt} = -D_v\bar{v} - D_\gamma\bar{\gamma} - (mg\cos\vartheta_1)\bar{\vartheta} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(MOTION IN VELOCITY DIRECTION)} \\
&(b) \; mv_1\frac{d\bar{\vartheta}}{dt} = L_v\bar{v} + L_\gamma\bar{\gamma} + L_{\delta_1}\bar{\delta}_1 + L_{\dot{\varphi}}\frac{d\bar{\varphi}}{dt} + (mg\sin\vartheta_1)\bar{\vartheta} \; \text{(LIFT)} \\
&(c) \quad I\frac{d^2\bar{\varphi}}{dt^2} = M_v\bar{v} + M_\gamma\bar{\gamma} + M_{\delta_1}\bar{\delta}_1 + M_{\dot{\varphi}}\frac{d\bar{\varphi}}{dt} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(PITCHING MOTION)} \\
&(d) \qquad \bar{\gamma} = \bar{\varphi} - \bar{\vartheta} \qquad\qquad \text{(ANGLE OF ATTACK DEFINITION)}
\end{aligned}\right\} \quad (3.23)$$

for small motions in the pitch plane. Although the analysis of the linear equations (3.23) is by far simpler than that of the corresponding general flight equations, such analysis has been very fruitful especially in connection with the following applications:

1. *Studies of aircraft stability.* The equilibrium motion of an aircraft is said to be *stable* if the aircraft tends to return to that equilibrium condition after a small disturbance due to aerodynamic disturbing forces or control deflections. In the case of stable motion, small changes in speed and attitude will tend to remain small; such a situation is frequently desirable.

2. *Studies of aircraft response to controls.* The linearized flight equations are frequently used for analyzing the response of different aircraft configurations to motions of the control surfaces. This may be accomplished by studying the transient behavior of the aircraft after a control surface deflection. Another particularly useful method for analyzing the sensitivity of the aircraft attitude to control motions applied at various rates is based on the study of steady-state sinusoidal variations of the flight variables. It is possible, for instance, to study the *amplitude and phase response* of the attitude angle changes $\bar{\varphi}$ and flight path angle $\bar{\vartheta}$ to applied sinusoidal changes in the elevator deflection at different frequencies. The amplitude response will indicate the sensitivity of the aircraft motion to control motions applied at different rates, whereas the phase response will show how quickly the aircraft can follow the control motion. Aerodynamic *transfer functions* like $\bar{\varphi}/\bar{\delta}_1$ or $\bar{\vartheta}/\bar{\delta}_1$ may be derived from the linearized flight equations (3.23) in much the same manner as voltage transfer functions are derived from electrical circuit equations.[1]

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[1] Seamans, R. C., B. G. Bromberg, and L. E. Payne, Application of the Performance Operator to Aircraft Automatic Control, *MIT DIC Rept.*, 1947 (Instrumentation Laboratory).

Figure 3.13*c* shows typical plots of the computed amplitude and phase response of the attitude angle φ of a B-25J aircraft[1] to small sinusoidal changes in the elevator deflection δ_1. The results of such computations can be applied to the study of more realistic maneuvers by means of Fourier analysis.

Finally, the study of small aircraft motions is important in connection with the design of automatic pilots (see Sec. 3.7).

D-c Analog-computer Solution of the Linearized Flight Equations. The solution of aircraft linearized flight equations has been one of the most fruitful applications of d-c analog computers. This is true not only because of the savings in time and labor effected but also because of the ease of eventually introducing the effects of nonlinearities such as stops limiting the control surface deflections (see Sec. 3.3) and of performing partial system tests.

A d-c analog computer set up to solve the linearized flight equations may be used to study the transient response of the aircraft with respect to small disturbances as well as the amplitude and phase response to sinusoidal disturbances of different frequencies. The amplitude and phase response may be measured exactly as shown in Sec. 3.3 for the case of servomechanisms simulated by the computer. D-c analog computers have become invaluable tools for the study of aircraft stability and control in an ever-increasing number of instances.

A typical d-c analog-computer setup for such purposes is described below in terms of the problem specifications and computer solution record forms for a typical case.

A. Problem specifications

1. *Introduction.* This paragraph will contain problem designations, authorization references, etc.
2. *Statement of the problem.* Set up a d-c analog computer to solve the linearized flight equations of an aircraft for small motions in the pitch plane. It is desired to compute variations in the speed, pitch attitude angle, and flight path angle of the aircraft as functions of the time for given initial conditions if the elevator deflection angle is given as a function of the time.
3. *Mathematical relations*

$$(a)\quad m\frac{d\bar v}{dt} = -D_v\bar v - D_\gamma\bar\gamma - (mg\cos\vartheta_1)\bar\vartheta$$

(MOTION IN VELOCITY DIRECTION)

$$(b)\quad mv_1\frac{d\bar\vartheta}{dt} = L_v\bar v + L_\gamma\bar\gamma + L_{\delta_1}\bar\delta_1 + L_{\dot\varphi}\frac{d\bar\varphi}{dt} + (mg\sin\vartheta_1)\bar\vartheta \qquad \text{(LIFT)}$$

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[1] Hagelbarger, Howe, and Howe, *op. cit.*

(*c*) $I\,\dfrac{d^2\bar{\varphi}}{dt^2} = M_v\bar{v} + M_\gamma\bar{\gamma} + M_{\delta_1}\bar{\delta}_1 + M_{\dot{\varphi}}\,\dfrac{d\bar{\varphi}}{dt}$ (PITCHING MOTION)

(*d*) $\bar{\gamma} = \bar{\varphi} - \bar{\vartheta}$ (ANGLE OF ATTACK DEFINITION)

These equations were derived under the following assumptions which flight tests have shown reasonably valid for aircraft of the type under consideration: (*a*) variables deviate from the equilibrium conditions by small increments only; (*b*) all coefficients remain constant throughout the maneuvers considered; (*c*) the coupling between the motion in the pitch plane (longitudinal motion) and the roll and yaw (lateral motion) is negligible; (*d*) the thrust remains constant throughout the maneuver.

4. *Glossary of symbols*

t	time
v_1	aircraft velocity v in equilibrium condition
ϑ_1	flight path angle ϑ in equilibrium condition
φ_1	pitch attitude angle φ in equilibrium condition
$\bar{v}$	change in velocity
$\bar{\vartheta}$	change in flight path angle
$\bar{\varphi}$	change in pitch attitude angle
$\bar{\gamma} = \bar{\varphi} - \bar{\vartheta}$	angle of attack change
$\bar{\delta}_1$	change in elevator deflection (upward)
g	acceleration of gravity
m	aircraft mass
I	pitch moment of inertia
$D(v,\gamma)$	drag
$L\left(v,\gamma,\delta_1,\dfrac{d\varphi}{dt}\right)$	lift
$M\left(v,\gamma,\delta_1,\dfrac{d\varphi}{dt}\right)$	pitch aerodynamic moment

The constant coefficients D_v, L_v, M_γ, M_δ, etc., are the partial derivatives of D, L, and M, respectively, with respect to the subscript variables; all these derivatives are obtained for the equilibrium value of v, and

$$\delta_1 = \frac{dv}{dt} = \frac{d\vartheta}{dt} = \frac{d\varphi}{dt} = \frac{d\gamma}{dt} = 0$$

5. *Initial conditions.* As specified for each particular problem within the ranges given below.

6. *Numerical values of constants*[1]

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[1] From *Operating Manual for the Goodyear Electronic Analog Computer*, Goodyear Aircraft Company, Akron, Ohio.

$$\frac{D_v}{m} = +0.02 \text{ ft/sec}^2$$

$$\frac{D_\gamma}{m} = +16.6 \text{ ft/sec}^2$$

$$g = 32.2 \text{ ft/sec}^2$$

$$\vartheta_1 = 0$$

$$v_1 = 270 \text{ ft/sec}$$

$$\frac{L_v}{m} = +0.237 \text{ ft/sec}^2$$

$$\frac{L_\gamma}{m} = +238 \text{ ft/sec}^2$$

$$\frac{L_{\delta_1}}{m} = -26.6 \text{ ft/sec}^2$$

$$\frac{L_{\dot{\varphi}}}{m} = +1.68 \text{ ft/sec}$$

$$\frac{M_v}{I} = 0$$

$$\frac{M_\gamma}{I} = -11.9 \frac{1}{\text{sec}^2}$$

$$\frac{M_{\delta_1}}{I} = +10.3 \frac{1}{\text{sec}^2}$$

$$\frac{M_{\dot{\varphi}}}{I} = -0.679 \frac{1}{\text{sec}^2}$$

7. *Estimated ranges of variables*

$0 \leq t < 50$ sec

$-25° < \bar{\vartheta} < 25°$

$-25° < \bar{\varphi} < 25°$

$-25° < \bar{\gamma} < 25°$

$-25° < \bar{\delta}_1 < 25°$

$-50 \text{ ft/sec} < \bar{v} < 50 \text{ ft/sec}$

$$25° = 0.436 \text{ radian}$$

8. *Parameters to be varied.* The effects of certain changes in aircraft configurations may be investigated by changes in the constants above.
9. *Expected results of parameter variations.* None noted here.[1]
10. *Possible analytical and numerical checks noted by the customer*
    a. A constant upward change $\bar{\delta}_1$ in the elevator displacement should be followed by an increase $\bar{\varphi}$ in the attitude angle. The resulting increase in the angle of attack γ will create additional lift which in turn will increase the flight path angle ϑ.
    b. The steady-state response of the aircraft to elevator deflections varying as sinusoidal functions of the time t may be determined analytically by replacing each operator d/dt in the linearized equations of motion by $j\omega$, where ω is the circular frequency of the sinusoidal oscillations.
11. *Use of results*
    a. To investigate small deviations from equilibrium flight conditions in the pitch plane.
    b. To investigate effects of changes in the aircraft configuration on aircraft stability in the pitch plane (longitudinal stability).

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[1] See, for instance, Von Mises, *op. cit.*

   c. Use of the flight simulator for testing both simulated and real automatic pilots (see Sec. 3.7).

12. *Accuracy required.* 0.5 to 1.5 per cent.

B. Record of computer solution

1. *Transformation equations.* All problem variables are measured in the ft/slug/sec system of units; all angles are measured in radians. All machine variables are measured in machine units. One machine unit equals 100 volts.

$t = \tau$ (1:1 time scale)

$\vartheta = 0.5\bar{\Theta} \qquad \bar{\varphi} = 0.5\bar{\Phi} \qquad \bar{\gamma} = 0.5\bar{\Gamma} \qquad \delta_1 = 0.5\bar{\Delta}_1$

$\bar{v} = 50\bar{V}$

2. *Machine equations*

(*a*) $P\bar{V} = -0.02\bar{V} - 0.166\bar{\Gamma} - 0.322\bar{\Theta}$ (MOTION IN VELOCITY DIRECTION)

(*b*) $P\bar{\Theta} = 0.0879\bar{V} + 0.882\bar{\Gamma} - 0.0986\bar{\Delta}_1 + 0.00622P\bar{\Phi}$ (LIFT)

(*c*) $P^2\bar{\Phi} = -11.9\bar{\Gamma} + 10.3\bar{\Delta}_1 - 0.679P\bar{\Phi}$ (PITCHING MOTION)

(*d*) $\bar{\Gamma} = \bar{\Phi} - \bar{\Theta}$ (ANGLE OF ATTACK DEFINITION)

3. *Machine initial conditions.* As specified for each particular problem.
4. *Block diagram* (see Fig. 3.12). The following types of function generators may be needed for generating $-\bar{\Delta}_1$ (simulated elevator deflection) as a function of the machine time τ.
   a. Potentiometer for manual operation.
   b. Step-function generator (voltage source and switch).
   c. Sine or cosine generator capable of continuous operation (see Sec. 6.5).
   d. Computer setups to generate (constant) $\cdot\, \tau$ and (constant) $\cdot\, \tau^2$ (see Sec. 3.3).
   e. Universal function generator (see Sec. 6.5).
5. *Discussion.* None.
6. *Enclosure* (see also Fig. 3.13). Dated and initialed computer records ($\bar{\delta}_1$, $\bar{v}$, $\bar{\gamma}$, $\bar{\varphi}$, $\bar{\vartheta}$, vs. t); plotted amplitude and phase characteristics of the "transfer function" $\bar{\varphi}/\bar{\delta}_1$ for sinusoidal elevator motion.

3.7. Application of D-c Analog Computers to the Design and Testing of Automatic Pilots. *Introduction.* A number of techniques for employing d-c analog computers as flight simulators for the design of automatic pilots will now be outlined in more detail. An automatic pilot is a system

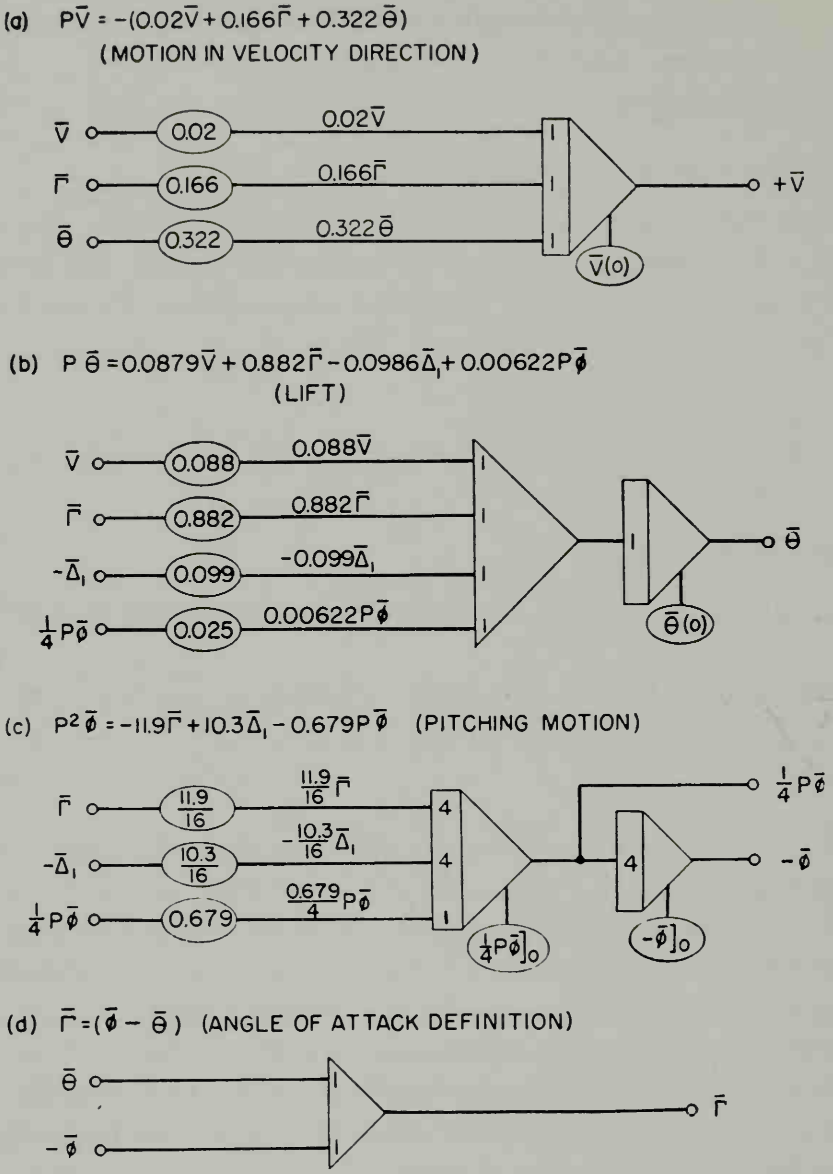


FIG. 3.12. Block diagram of a flight-simulator setup for the study of small aircraft motions in the pitch plane. Note how each equation of motion is represented by a block of computing elements. The simulated "aircraft" may be "steered" by varying the input voltage $-\bar{\Delta}$, which corresponds to a change in the elevator deflection.

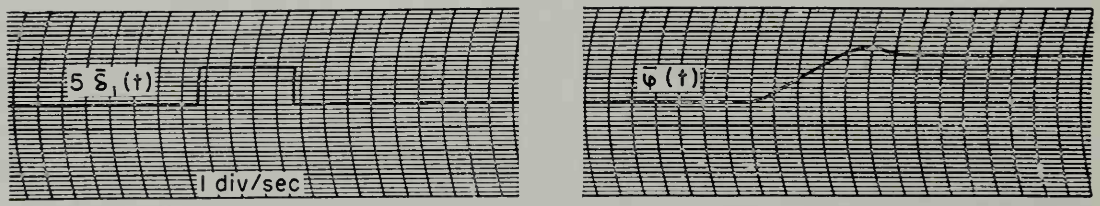
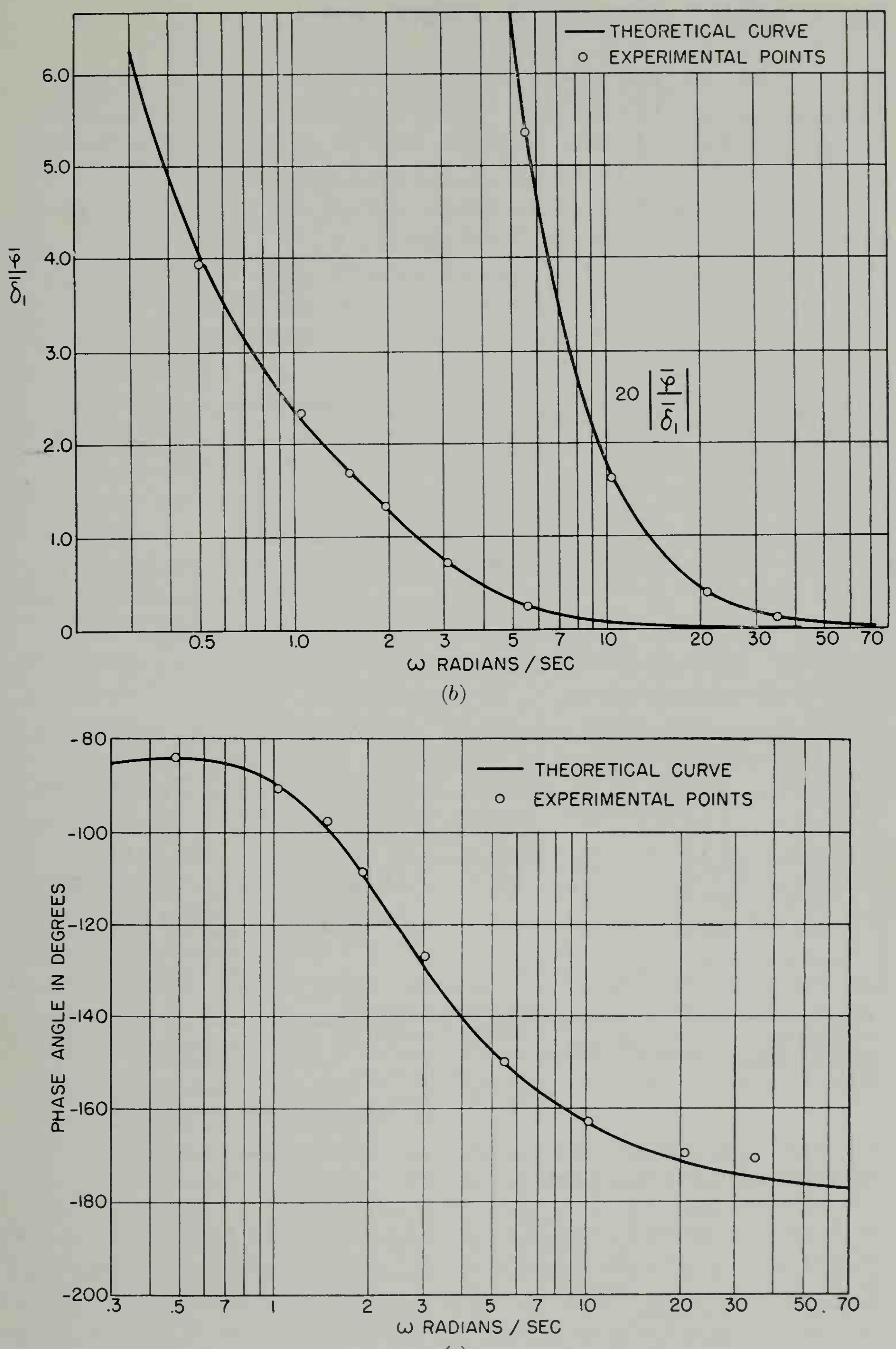


FIG. 3.13*a*. Flight-simulator record (from Hagelbarger, Howe, and Howe, *op. cit.*) of the attitude change of an aircraft in response to an elevator deflection. The solutions of Fig. 3.13*a* to *c* are not for the aircraft configuration discussed in the text.



(*b*)

(*c*)

FIG. 3.13*b* and *c*. Experimental and calculated amplitude and phase response of the aircraft attitude change $\bar{\varphi}$ to sinusoidal elevator deflections (From Hagelbarger, Howe, and Howe, *op. cit.*). The solutions of Fig. 3.13*a* to *c* are not for the aircraft configuration discussed in the text.

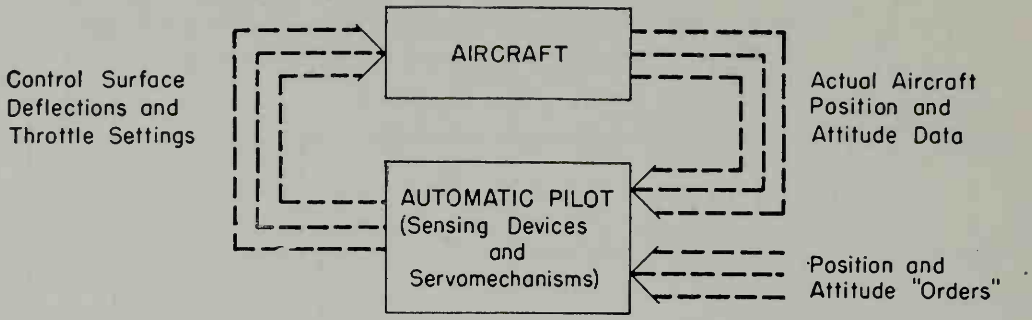

Control Surface Deflections and Throttle Settings
AIRCRAFT
Actual Aircraft Position and Attitude Data
AUTOMATIC PILOT (Sensing Devices and Servomechanisms)
Position and Attitude "Orders"

FIG. 3.14*a*. Block-diagram representation of autopilot operation.

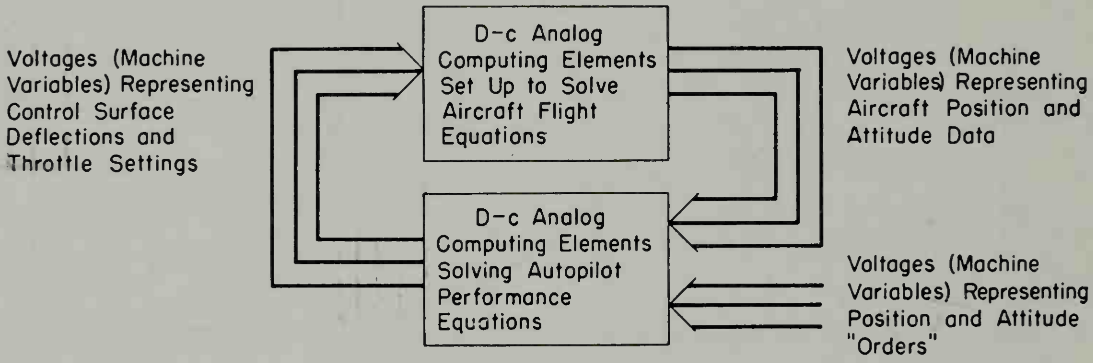

Voltages (Machine Variables) Representing Control Surface Deflections and Throttle Settings
D-c Analog Computing Elements Set Up to Solve Aircraft Flight Equations
Voltages (Machine Variables) Representing Aircraft Position and Attitude Data
D-c Analog Computing Elements Solving Autopilot Performance Equations
Voltages (Machine Variables) Representing Position and Attitude "Orders"

FIG. 3.14*b*. General type of d-c analog-computer setup for simulating the operation of an aircraft-autopilot system.

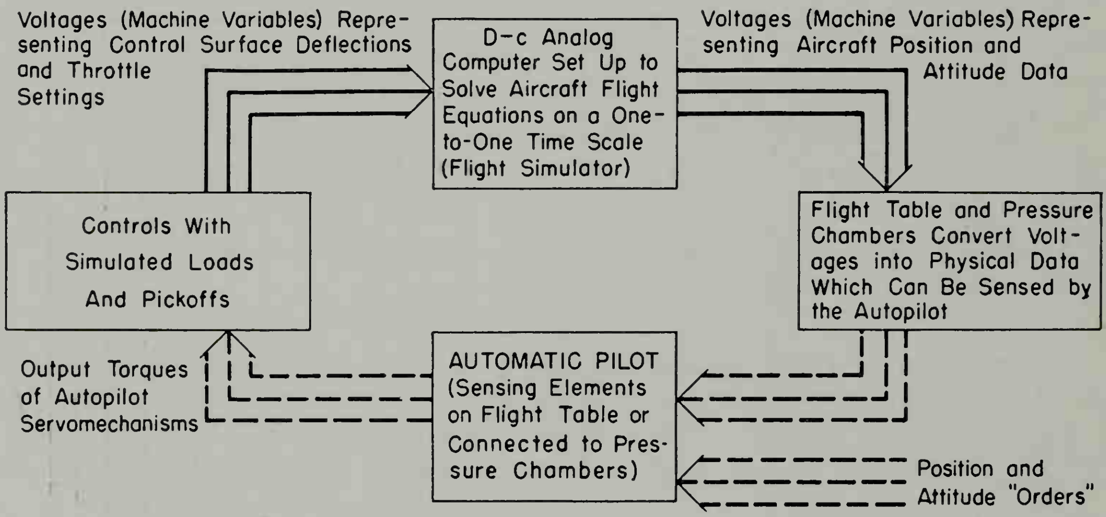

Voltages (Machine Variables) Representing Control Surface Deflections and Throttle Settings
D-c Analog Computer Set Up to Solve Aircraft Flight Equations on a One-to-One Time Scale (Flight Simulator)
Voltages (Machine Variables) Representing Aircraft Position and Attitude Data
Controls With Simulated Loads And Pickoffs
Flight Table and Pressure Chambers Convert Voltages into Physical Data Which Can Be Sensed by the Autopilot
Output Torques of Autopilot Servomechanisms
AUTOMATIC PILOT (Sensing Elements on Flight Table or Connected to Pressure Chambers)
Position and Attitude "Orders"

FIG. 3.14*c*. Block diagram showing how a d-c analog computer may be used as a flight simulator for partial system testing.

of servomechanisms designed to adjust the control surface deflections and possibly the throttle settings of an aircraft continuously or intermittently. The automatic pilot will operate in such a manner that all or some of the coordinates describing the aircraft attitude and position and/or their derivatives correspond as closely as possible to certain desired "reference" values which may be fixed or vary as functions of the time. Automatic pilots used for *attitude control* will position control surfaces so as to maintain a desired attitude in roll, pitch, and yaw or so as to produce desired rates of turn or rates of climb or descent. Automatic pilots may also achieve *altitude control* and *speed control*, as well as control of the *angle of attack.* Automatic control devices intended primarily for *trajectory control* are not usually referred to as automatic pilots but may be studied in much the same manner. Devices in this latter category include equipment used for blind landings as well as long-range-navigation, homing, and guidance apparatus. They will usually comprise a computer which translates position data from some type of navigational device into orders for an automatic pilot which regulates control surface deflections and/or throttle settings.

Automatic-pilot Operation. The nature of automatic-pilot operation as well as the great importance of analog computers in the development of automatic pilots can perhaps be understood best after an inspection of the block-diagram representation of an automatic-pilot system shown in Fig. 3.14*a*.

In this block diagram, the aircraft is represented as a "transducer" whose input data are instantaneous values of control surface deflections and throttle settings. The output data are the instantaneous values of the coordinates specifying the aircraft attitude and position. The automatic pilot, in turn, measures the attitude and position "errors," or differences, between the actual and desired values of the attitude and position coordinates by means of gyros, accelerometers, pressure-sensitive devices, etc. These data are reduced to input signals for the servomechanisms which position the control surfaces and throttles. The latter are made to control the aircraft attitude and position so as to minimize the attitude and position errors.

In Fig. 3.14*a*, the automatic pilot, like the aircraft, is represented as a "transducer"; its "input signals" are the aircraft attitude and position data as well as the preset orders, and its output data are control surface deflections and throttle settings. Together the aircraft and the automatic pilot constitute a *feedback loop.*

A properly designed automatic pilot for a certain aircraft must control that aircraft in accordance with its "orders" to within a specified error range and without any tendency toward instability or uncontrolled oscillations.

Since the automatic pilot comprises a number of servomechanisms, it will itself contain a number of internal feedback loops used to obtain the correct control surface deflections and throttle settings as functions of the input data. The entire system consisting of the automatic pilot and the aircraft is, then, a rather complex multiloop feedback system whose analysis by ordinary means may become quite complicated, especially if the performance of system components cannot be described in terms of linear differential equations.

Simulation of the Aircraft–Automatic-pilot System. Figure 3.14*b* illustrates the use of a d-c analog computer as a flight simulator for automatic-pilot design. The performance of the aircraft is represented by a block of computing elements set up for solution of the flight equations. This block of computing elements accepts voltages proportional to the control surface deflections and throttle settings; output voltages proportional to the corresponding attitude and position coordinates of the aircraft are obtained through continuous integration of the flight equations. The automatic pilot may, in turn, be represented by a block of computing elements which accept attitude and position data as input voltages and continuously compute output voltages proportional to the control surface deflections and throttle settings produced by the automatic pilot. *The entire feedback loop of Fig.* 3.14*a is thus "simulated" by the "analogous" computer setup of Fig.* 3.14*b.*

Figure 3.14*c* shows schematically how a *partial system test* of an automatic pilot may be performed through the use of a d-c analog computer set up as a flight simulator operating on a 1:1 time scale. The output voltages of the flight simulator are used to position a *flight table* and, if desired, to vary the pressures in small pressure chambers in accordance with the simulated aircraft motion. The sensing elements of an actual automatic pilot under test are used to detect these changes and will cause the servomechanisms of the automatic pilot to position control surfaces and throttles loaded artificially by springs, motors, or solenoids.

Pickoff potentiometers permit one to obtain voltages proportional to the control surface deflections and throttle settings; these voltages, in turn, become the input voltages of the flight simulator, so that the entire control loop is, again, simulated; a setup of this type permits very realistic testing of automatic-pilot performance.

A Practical Design Procedure. The design of a projected automatic pilot for a given aircraft may proceed in a series of steps constituting successive approximations to the realistic simulation of actual flight conditions. Such a stepwise procedure will permit the designer to introduce himself gradually to the difficulties involved in ascertaining the performance of a complicated automatic control system.

1. The flight equations of the aircraft under consideration are set up on the d-c analog computer, and the resulting flight simulator setup is checked.

2. Frequently the most effective approach to control-system design is through an initial study of a greatly simplified system. Thus each servomechanism of a projected automatic pilot may be initially represented by

a perfect proportional follow-up device having a certain constant gain. This amounts to the assumption of a perfect servomechanism which can translate any order into a corresponding proportional control surface deflection or change in throttle setting.

3. As the simulated servomechanism gains in such a computer setup are raised in order to obtain sufficiently low attitude and position errors ("tight" control), "hunting" or instability will usually result. The next step in the design of each automatic-pilot servomechanism may be to introduce equalizing networks (rate feedback, integral feedback, or even nonlinear types of feedback) into the control loop while still assuming "perfect" servomechanisms having unlimited amplitude and frequency response.

4. After a preliminary choice of the equalization devices has been made in this manner, it becomes necessary to make a choice of actual error-sensing devices and servomotors which most closely approach the desired ideal characteristics within the working frequency range of the automatic pilot while still satisfying given space, weight, and power requirements.

It will usually be found that especially the high-frequency response of actual servomotors does not correspond to the idealized conditions considered in step 3.

The equations of motion of the actual automatic-pilot components are next included in the d-c analog-computer setup. In other words, each servomechanism of the automatic pilot is represented by d-c analog computing elements in the manner discussed in Sec. 3.3 in as great detail as is found necessary. The design parameters of the various servomechanisms and equalization networks may then be varied until the best possible response is approximated. The values of the parameters so obtained will serve as basic design data for the automatic pilot.

5. After a model of the actual automatic pilot has been assembled according to these considerations, it may be connected together with the flight simulator for a *partial system test* during which further adjustments can be made.

6. The equipment is then packaged for an actual flight test, is given the usual drop and vibration tests, and is checked out in flight. Although the flight test must always be the final criterion of the suitability of an automatic-pilot design, a large amount of actual flight time may be saved by the procedure outlined above.

The d-c analog-computer setup for the automatic-pilot development outlined above may be operated on an expanded time scale if this is desirable; only the flight simulator used for partial system testing must necessarily be operated on a 1:1 time scale. It may, however, often be useful to utilize the same flight simulator for both partial system testing and simulation of the automatic pilot, since by convenient alternation of the two methods, it is possible to make rapid checks on proposed changes in the automatic-pilot design.

It will usually be advantageous to utilize both the frequency analysis and transient analysis methods discussed in Sec. 3.3 *throughout the automatic-pilot development procedure outlined above.* The use of the d-c analog computer also makes it possible to introduce nonlinear characteristics of both aircraft and servomechanisms, such as torque and speed limitations of motors, stops limiting control surface deflections, and backlash in gear trains (see Sec. 3.3).

Flight Simulator Study of Automatic-pilot Performance for Small Motions in the Pitch Plane. The use of a d-c analog computer for the design and testing of a simple automatic pilot intended to control aircraft motions in the pitch plane (see Sec. 3.6) is an interesting application of the principles outlined above. The proposed automatic pilot is to "sense" the pitching motion of a certain aircraft by means of free and rate gyros and to control the elevator deflections so as to keep the pitch attitude angle as closely as possible equal to a reference attitude set into the automatic pilot. If the reference attitude angle is constant, the automatic pilot will attempt to maintain a constant aircraft attitude φ equal to the preset reference attitude. If the reference attitude is "programmed" as a function of time, the automatic pilot will attempt to make the aircraft attitude follow the reference.

Block diagrams of d-c analog-computer setups used in connection with the development of such an automatic pilot are shown in Fig. 3.15 *a* to *e*. Since conventional automatic pilots are, in general, not used during violent maneuvers, it will be sufficient to study small motions of the aircraft about an equilibrium condition. For the sake of simplicity, only motions in the pitch plane are considered, as in Sec. 3.6. The equivalent three-dimensional problem is, again, no more difficult in principle, but the resulting computations are more complex.

The design parameters of any automatic pilot must depend strongly on the design of the aircraft to be controlled, since the response of different aircraft to similar control surface deflections may be entirely different. Assuming that the aircraft under consideration is of a type similar to that discussed in Sec. 3.6, a flight simulator of the type shown in Fig. 3.15*a* may be used to compute the attitude changes of the aircraft in response to applied changes in the elevator deflection. The machine equations and the computer setup of the flight simulator shown are those derived in Sec. 3.6; the effects of reasonable changes in aircraft configuration (changes in the areas of wing, stabilizer, or elevator surfaces; length of the fuselage; etc.) or equilibrium speed may be introduced by changing coefficient settings (see Sec. 3.6). The flight simulator will accept an input voltage proportional to the elevator deflection changes $\bar{\delta}_1$ and will continuously compute output voltages proportional to the pitch attitude angle change $\bar{\varphi}$ and the pitch rate $d\bar{\varphi}/dt$ on a 1:1 time scale. A real or simulated automatic pilot may then be made to "sense" the simulated pitching motion and can, in turn, be made to "control" the simulated elevator deflections, so that the performance of the entire control loop can be studied.

First Approximation to the Performance of an Actual Automatic Pilot. Figure 3.15*b* shows a d-c analog-computer setup simulating the performance of a hypothetical automatic pilot which makes the elevator surface deflection $\bar{\delta}_1$ equal and opposite to the weighted sum

$$a_1\left(k_1\frac{d\bar{\varphi}}{dt} + \bar{\varphi} - \bar{\varphi}_r\right) = a_1[(k_1p + 1)\bar{\varphi} - \bar{\varphi}_r]$$

Such an automatic pilot would have to incorporate a "perfect" servomechanism capable of positioning the control surface *instantaneously* in accordance with the input signals supplied from the free gyro, which "senses" the attitude error $(\bar{\varphi}_r - \bar{\varphi})$, and the rate gyro, which senses the pitch rate $d\bar{\varphi}/dt$. The computer setup of Fig. 3.15*b* may be said to constitute a first approximation to the performance of an actual automatic pilot.

Second Approximation to the Performance of an Actual Automatic Pilot. In an actual automatic pilot, the control surface deflection cannot follow the input signals instantaneously, mainly because of the inertia, spring loads, and friction associated with the mechanical components of the system. As a second approximation to the performance of an actual automatic pilot, the transfer function of the automatic-pilot servomechanism may be approximated by the simple expression

$$\frac{\text{constant}}{bp + 1} \tag{3.24}$$

so that

$$\bar{\delta}_1 = -\frac{a_1}{bp + 1}[k_1p\bar{\varphi} + (\bar{\varphi} - \bar{\varphi}_r)] \tag{3.25}$$

The somewhat crude approximation (3.24) is frequently useful for investigating the approximate operation of automatic control mechanisms. Using the transformation equations

$$t = \tau \qquad \bar{\varphi} = 0.5\bar{\Phi} \qquad \bar{\varphi}_r = 0.5\bar{\Phi}_r \qquad \bar{\delta}_1 = 0.5\bar{\Delta}_1 \tag{3.26}$$

corresponding to those used in Sec. 3.6, the machine equation correspond-

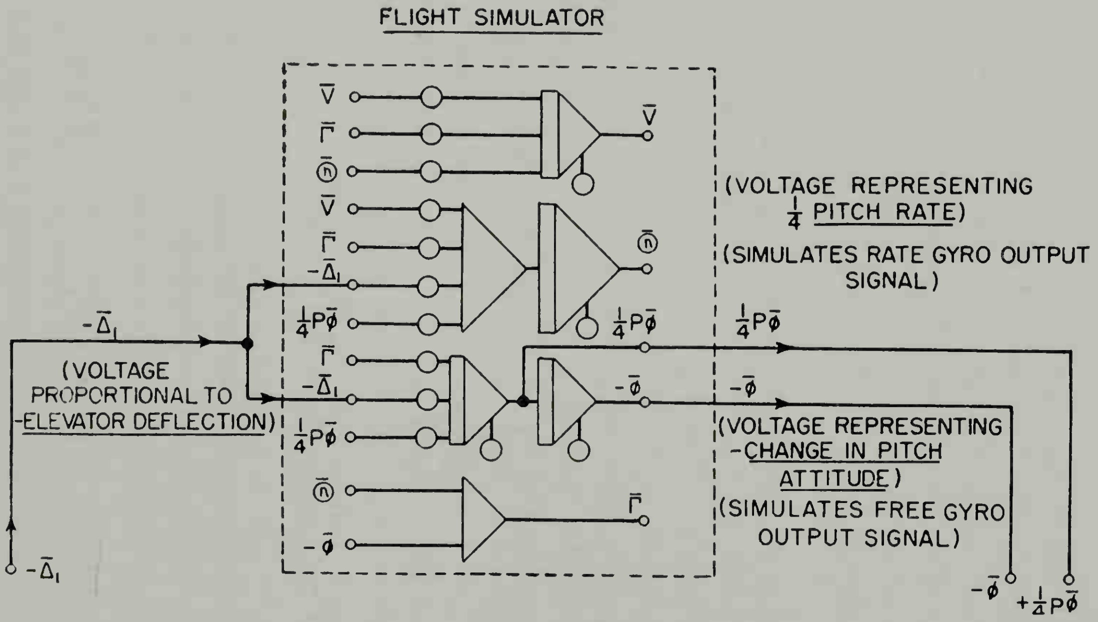


FIG. 3.15*a*. Flight-simulator setup for studying the performance of simulated or real automatic pilots for motion in the pitch plane. The flight simulator computes voltages proportional to pitch attitude changes and pitch rate continuously by solving the flight equations for small deviations from equilibrium in the pitch plane (see Sec. 3.6).

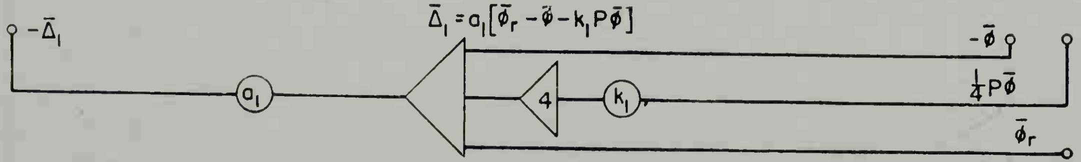


FIG. 3.15*b*. Computer setup simulating the performance of an automatic pilot with perfect servomechanism (first approximation to the performance of a real autopilot).

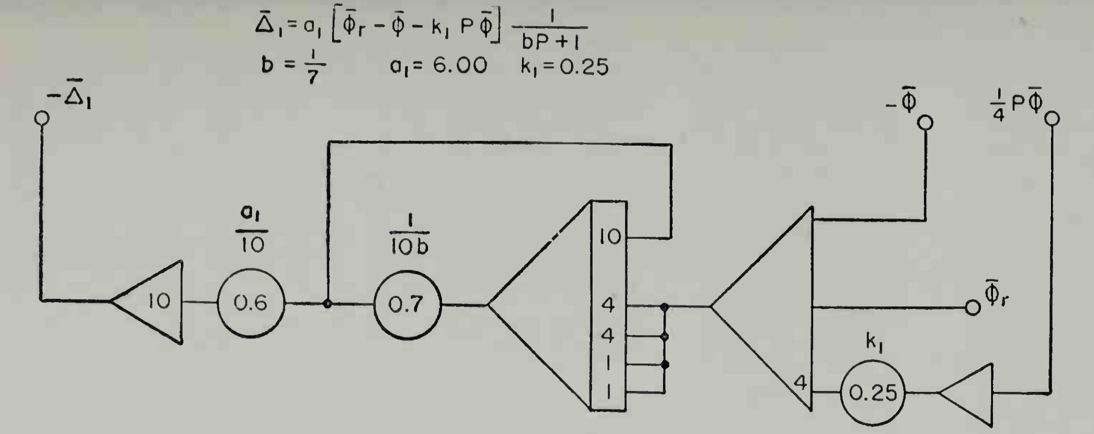

$\bar{\Delta}_1 = a_1 \left[\bar{\Phi}_r - \bar{\Phi} - k_1 P \bar{\Phi}\right] \frac{1}{bP+1}$
$b = \frac{1}{7}$ $a_1 = 6.00$ $k_1 = 0.25$
$-\bar{\Delta}_1$
10
$\frac{a_1}{10}$
0.6
$\frac{1}{10b}$
0.7
10
4
4
1
1
4
$-\bar{\Phi}$
$\bar{\Phi}_r$
k_1
0.25
$\frac{1}{4}P\bar{\Phi}$

FIG. 3.15*c*. Computer setup simulating the performance of an automatic pilot by means of a simple time-delay circuit (second approximation to the performance of a real autopilot).

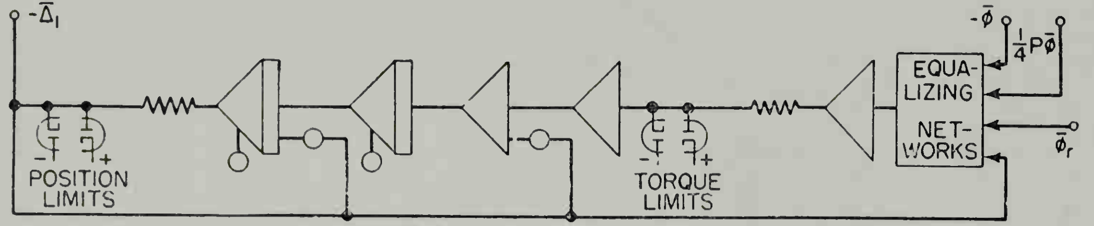


FIG. 3.15*d*. Computer setup simulating the performance of an automatic pilot having an electric motor and equalizing networks as well as position and torque limits (third approximation to the performance of a real autopilot). Note that the simulation of the equalizing networks will involve the use of a phase-inverting amplifier in the case shown.

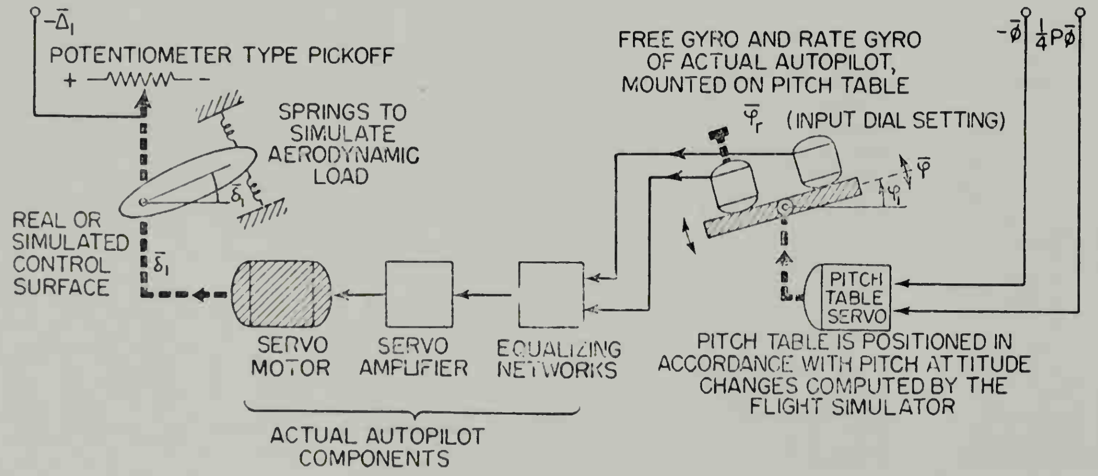

$-\bar{\Delta}_1$
POTENTIOMETER TYPE PICKOFF
SPRINGS TO SIMULATE AERODYNAMIC LOAD
δ_1
REAL OR SIMULATED CONTROL SURFACE
$\bar{\delta}_1$
SERVO MOTOR
SERVO AMPLIFIER
EQUALIZING NETWORKS
ACTUAL AUTOPILOT COMPONENTS
FREE GYRO AND RATE GYRO OF ACTUAL AUTOPILOT, MOUNTED ON PITCH TABLE
$\bar{\varphi}_r$ (INPUT DIAL SETTING)
$\bar{\varphi}$
φ_1
$-\bar{\phi}$
$\frac{1}{4}P\bar{\phi}$
PITCH TABLE SERVO
PITCH TABLE IS POSITIONED IN ACCORDANCE WITH PITCH ATTITUDE CHANGES COMPUTED BY THE FLIGHT SIMULATOR

FIG. 3.15*e*. Flight-table setup for testing actual automatic-pilot components with the flight simulator of Fig. 3.15*a*. The servo feedback connections are not shown.

⟵ ELECTRICAL CONNECTION ‑ ‑ ⟵ ‑ ‑ MECHANICAL CONNECTION

FIG. 3.15*a* to *e*. D-c analog-computer setups for the increasingly detailed simulation of an automatic pilot under study.

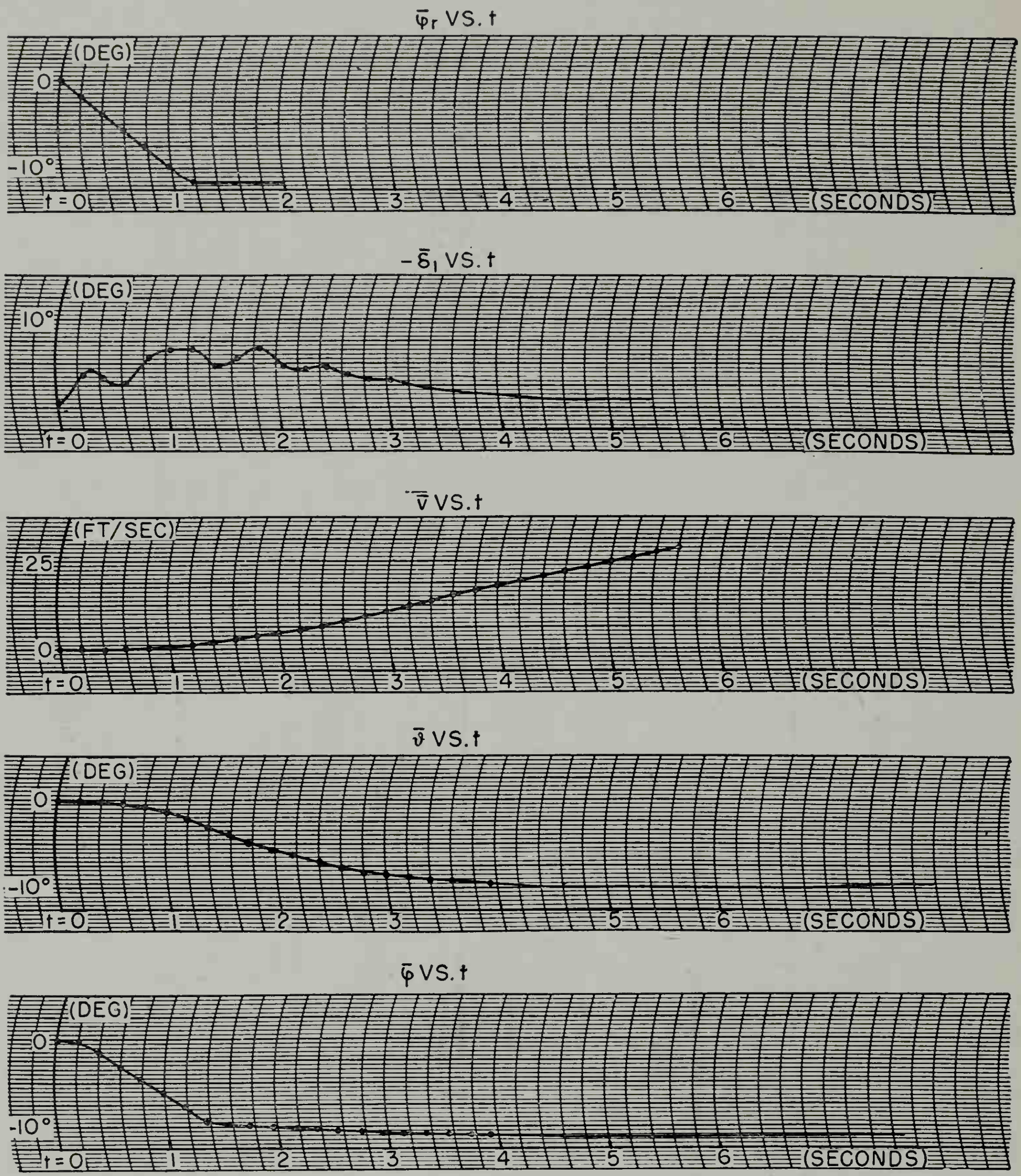


FIG. 3.16. Computer record for a typical "push-over" maneuver controlled by an automatic pilot (Goodyear Aircraft Company, *loc. cit.*). A multichannel recorder was used to present five variables plotted on the same time scale.

ing to (3.25) becomes

$$bP\bar{\Delta}_1 = -[a_1(\bar{\Phi} - \bar{\Phi}_r) + a_1k_1P\bar{\Phi} + \bar{\Delta}_1] \tag{3.27}$$

The computer setup shown in Fig. 3.15*c* establishes the machine equation (3.27) for

$$a_1 = 6 \qquad k_1 = 0.25 \text{ sec} \qquad b = 0.143 \text{ sec}$$

Figure 3.16 shows a set of typical computer records for an automatic-pilot-controlled maneuver specified by giving the automatic-pilot reference setting $\bar{\varphi}_r$ as a function of the time t.

Third Approximation to the Performance of an Actual Automatic Pilot. In the computer setup of Fig. 3.15*d*, the performance characteristics of an actual automatic pilot are simulated even more closely by computing elements capable of solving the performance equation of the automatic-pilot servomechanism. A servomechanism comprising special equalizing networks, an electric motor, and a spring-inertia load has been simulated in the manner described in Sec. 3.3. In addition, inelastic stops on the control surface and the torque limitations of the servomotor are simulated by pairs of parallel diode limiters at the correct points of the computer setup of Fig. 3.15*d*. All important design parameters of the simulated automatic pilot can be varied by changes in potentiometer settings or changes in the computer setup.

There is almost no limit on the refinements possible in simulator setups of this kind. Among other interesting variations one may introduce the effects of motor-field time delays, dry friction and gear backlash (see Sec. 3.3), disturbing torques on the aircraft or the control surfaces, dead spaces in the gyro response (see Sec. 3.3), and errors in the gyro output signals. The simple electric motor simulated in Fig. 3.15*d* could also be replaced by computer setups simulating clutch servos, hydraulic actuators, etc.

The results of parameter studies of automatic-pilot-controlled aircraft can be evaluated exactly in the manner shown for less complex servomechanisms in Sec. 3.3. As a further example, Fig. 3.17 shows the transient response of another automatic-pilot-controlled aircraft. The effect on the transient response of changing the value of the automatic-pilot gain is shown. Figure 3.18 shows the amplitude and phase of the transfer function $\bar{\varphi}/\bar{\varphi}_r$ for steady-state sinusoidal oscillations of a simulated automatic-pilot-controlled aircraft plotted vs. the circular frequency ω.

Flight Simulator Setup for Partial System Testing of Actual Automatic-pilot Components. Figure 3.15*e* shows how the flight simulator setup of Fig. 3.15*a* may be used for testing actual automatic-pilot components. The "output voltages" $-\bar{\Phi}$ and $\frac{1}{4}P\bar{\Phi}$ of the flight simulator are fed to a servomechanism which positions a *flight table* (pitch table). The angle between the pitch-table surface and the horizontal is thus kept equal to the pitch attitude angle $\varphi = \varphi_1 + \bar{\varphi}$ of the simulated aircraft as the attitude change $\bar{\varphi}$ about the equilibrium attitude φ_1 varies in accordance with the flight simulator output voltage $-\bar{\Phi}$. The free and rate gyros of the automatic pilot under test are mounted on the pitch table, and their output signals cause the actual automatic-pilot elevator servo to position an actual or simulated elevator surface. The latter is loaded by springs simulating the aerodynamic forces on the control surface for each aircraft speed under consideration. A potentiometer-type pickoff on the control surface shaft is used to generate a voltage $-\bar{\Delta}_1$ proportional to the control surface displacement $\bar{\delta}_1$ and representing the latter in the flight simulator.

Utilization of Computer Servomechanisms in Flight Simulators. Flight simulators intended for partial system testing of more complicated automatic pilots, such as those used for preventing stalls, may require the use of computer servomechanisms for solving nonlinear flight equations (see Sec. 3.5). Since a d-c analog computer used as a flight simulator must operate on a 1:1 time scale, it may be necessary to consider the possible effects of limitations in the high-frequency response of the computer servomechanisms used. In this connection, it is useful to remember that the frequency-response limitations of computer servomechanisms apply only to one of the two machine variables to be multiplied. It is fre-

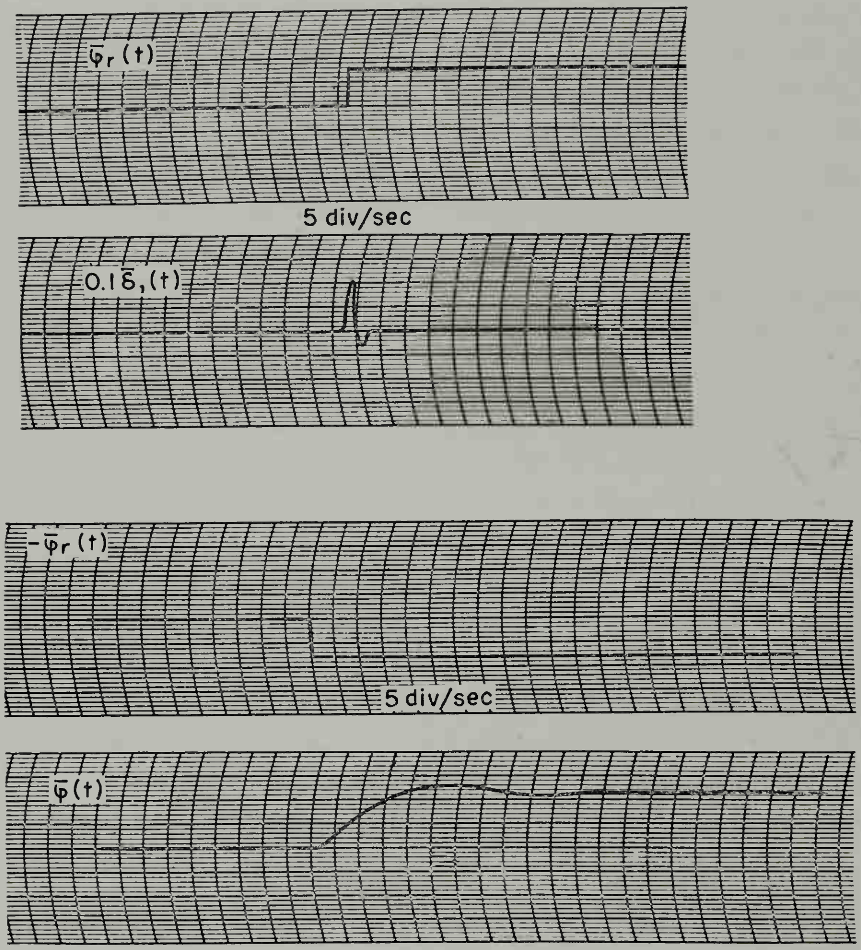


FIG. 3.17*a*. D-c analog-computer records (Hagelbarger, Howe, and Howe, *op. cit.*) showing the transient response of the elevator deflection $\bar{\delta}_1$ and of the pitch attitude change $\bar{\varphi}$ of a simulated autopilot-controlled aircraft to a step-function change in the reference attitude $\bar{\varphi}_r$.

quently possible to choose a problem variable which varies slowly to correspond to the machine variable appearing as a servo shaft displacement in the computer. Thus, for most flight simulators, the inertia associated with aircraft attitude and position changes is, as a rule, by far larger than the inertia associated with the rotations of computer servomechanism shafts whose displacements are to simulate the attitude and position changes of the aircraft. This usually applies even to the fairly large servomechanisms positioning the flight tables used for simulating aircraft rotations in the laboratory.

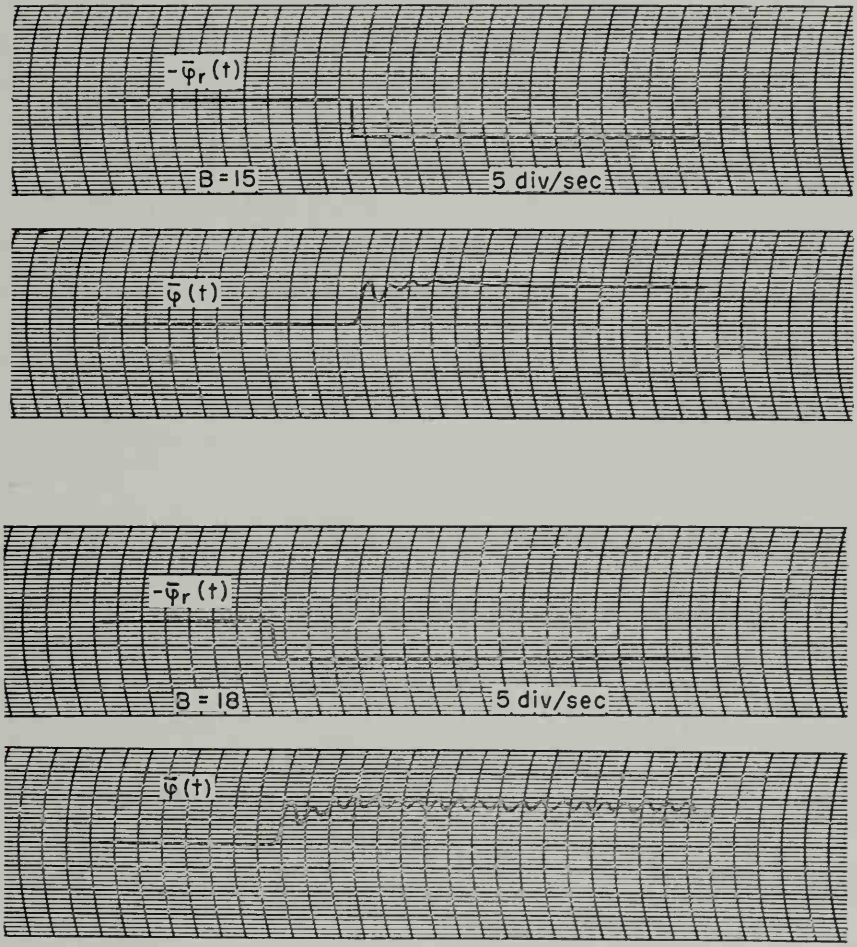


Fig. 3.17*b*. D-c analog-computer records (Hagelbarger, Howe, and Howe, *op. cit.*) showing the effect of increasing the forward gain of the simulated autopilot used to obtain the computer records of Fig. 3.17*a* by a factor of five and six, respectively.

3.8. Miscellaneous Applications. *Introduction.* The preceding sections illustrate the solution of representative problems by means of electronic differential equation solvers of the d-c analog type. The following list of problems involving the solution of sets of ordinary differential equations may suggest other applications of such machines.

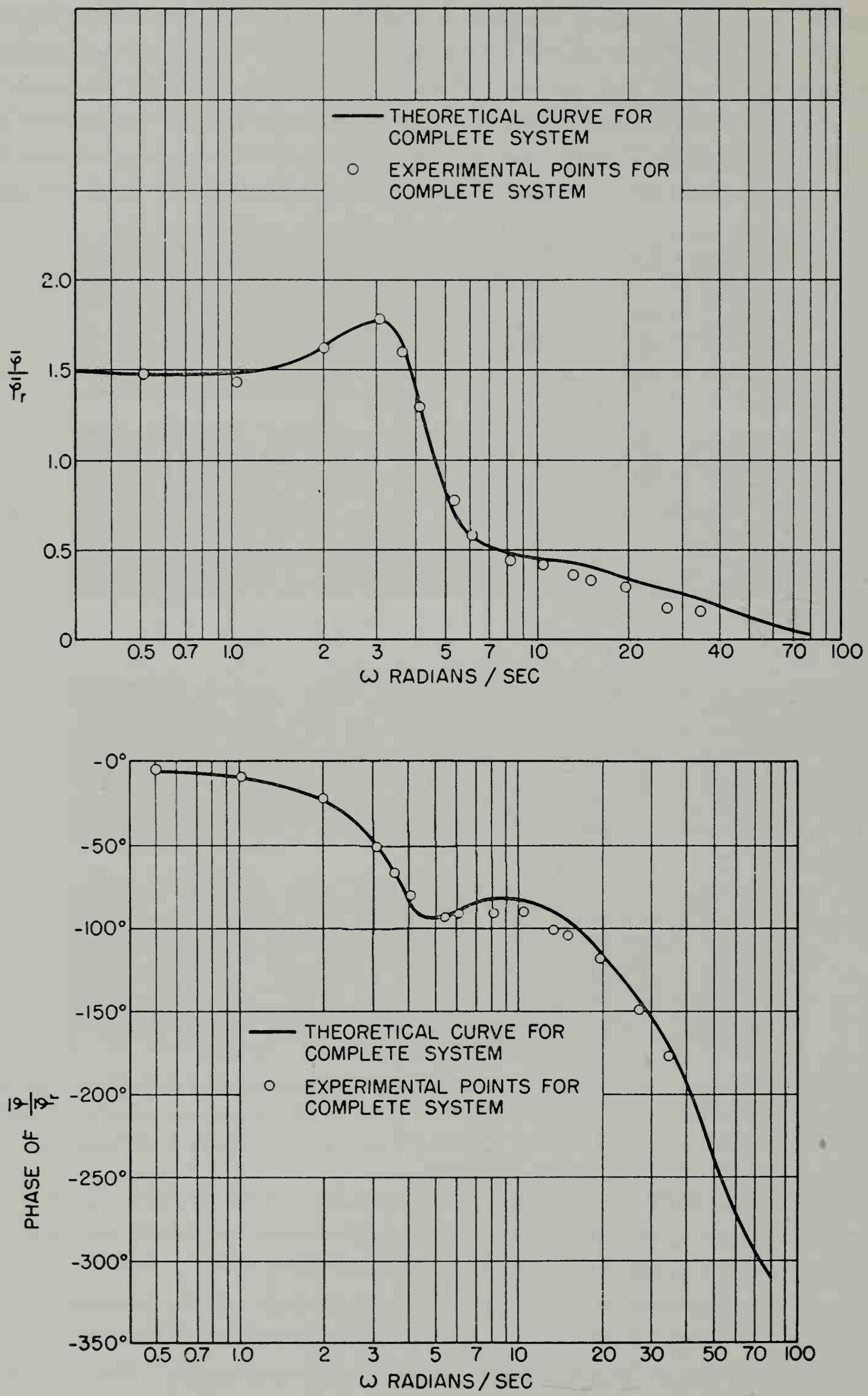


FIG. 3.18. Amplitude and phase response of an autopilot-controlled aircraft simulated by d-c analog computing elements (Hagelbarger, Howe, and Howe, *op. cit.*)

1. *Aircraft engineering*
    Design of aircraft controls (see Secs. 3.5 and 3.6)
    Design of automatic pilots (see Sec. 3.7)
    Turbojet and supercharger controllers
    Helicopter vibrations and control
    Vibrations and nonstationary flutter
    Gust loads
    Design of guidance and landing systems
    Landing shocks
    Rocket burning and flight
    Trajectory computations (see Secs. 3.4 and 3.5)
    Design of wind-tunnel mounting systems
    Aerodynamic stability of aerial refueling devices
    Supersonic flow problems
    Partial system testing
    Partial system testing involving human operators
2. *Mechanical and automotive engineering*
    Design and partial system testing of automatic control systems (see Sec. 3.3)
    Production testing of control devices
    Vibrations of structures
    Spring systems and vibration absorbers (see Sec. 3.2)
    Torsional vibrations in crankshafts
    Dynamics of mechanisms
    Hydraulic transmissions
    Flow problems
3. *Electrical engineering*
    Analysis of linear and nonlinear circuits
    Studies of modulation systems
    Studies of the effects of noise in communications systems
    Electron optics
    Feedback amplifiers and automatic control systems
    Analysis of rotating machinery
    Design of loudspeaker systems and enclosures
    Analysis of transmission lines
4. *Others*
    Process control in chemical plants
    Fire control
    Ballistic trajectories (see Sec. 3.4)
    Solutions of Schrödinger's equation in atomic physics

This list is by no means a complete one, and many other problems can, no doubt, be added.

The remainder of this section will be devoted to computer applications which do not directly involve the solution of differential equations; in addition, a brief discussion of the solution of differential equations involving boundary and eigenvalue problems is presented.

Automatic Equation Solvers. The solution of a set of machine equations like

$$\left.\begin{aligned} X + Y &= 0 \\ X - Y &= 2 \end{aligned}\right\} \tag{3.28}$$

for the unknown voltages X and Y by means of an analog computer is a problem different from that of establishing the machine equation

$$X + Y = 0 \tag{3.29}$$

simultaneously with a differential equation like

$$\frac{dX}{d\tau} - Y = 0 \tag{3.30}$$

by means of an analog machine. Since the differential equation (3.30) is essentially a relation involving only *changes* of the variable X, the machine equations (3.29) and (3.30) can be satisfied *continuously* for any reasonable set of initial values of X. This is *not* true for the machine equations (3.28). Correctly designed automatic equation solvers may enforce relations like (3.28) only *after an initial transient.* A machine of this type has actually solved a set of differential equations, *viz.*, the performance equations of the amplifiers, servomechanisms, and networks used for the computation. If these are correctly designed for the purpose, the machine variables X and Y will be made to approach the correct values corresponding to the solution of the given set of equations in the course of the "solution" of the performance equations. An *automatic equation solver* may thus be an analog computer comprising amplifiers and/or servomechanisms set up so that its performance equations cause the machine variables to approach the correct unknown values without any tendency toward *instability* or uncontrolled oscillations. *If* the machine variables finally assume constant values within the dynamic range of the computing elements, these values will belong to the set of the correct unknowns. Generally applicable analytical conditions for the stable performance of such machines have been formulated for the case of linear equation solvers,[1] but more remains to be done in the general case.[2]

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[1] Korn, G. A., Stabilization of Simultaneous Equation Solvers, *Proc. IRE*, **37**: 1000, 1949.

[2] Murray, Francis J., *The Theory of Mathematical Machines*, King's Crown Press, New York, 1947.

Utilization of D-c Analog Computers as Automatic Equation Solvers. Existing d-c analog computers are usually intended mainly for the solution of sets of differential equations. In machines specifically intended to solve problems involving neither differentiations nor integrations, the utilization of a-c amplitudes as machine variables may frequently be preferable to the use of d-c analog computing techniques; the use of a-c amplifiers will eliminate drift as an error source. Multipurpose d-c analog computers can, nevertheless, be quite useful for the solution of certain types of equations and sets of equations.

The solution of a single equation like

$$f(x) = 0$$

for the unknown x must be translated into the solution of a corresponding machine equation

$$F(X) = 0 \qquad (3.31)$$

for an unknown voltage X. Automatic equation solvers may be set up with ordinary d-c analog computing elements by one of the following methods:

1. *Solution by implicit computation.* This method was introduced in Sec. 1.2. The machine equation (3.31) is replaced by an equivalent equation

$$F(X) + X = X \qquad \text{or} \qquad X - F(X) = X \qquad (3.32)$$

and this relation is set up on the computer in the usual manner. This is shown in Fig. 3.19*a* for the general case as well as for a simple linear equation. The stability of the resulting computer setup must be investigated in each case.

2. *Use of high-gain amplifiers.* This method, illustrated in the solid lines of Fig. 3.19*b*, is discussed in Sec. 4.6. High-gain servomechanisms may be used in the same manner. The stability of the resulting computer setups must be investigated in each case.

3. *Use of integrators as high-gain amplifiers.* Since a d-c integrator has a very high gain at zero frequency, ordinary d-c integrators may replace high-gain amplifiers in automatic equation-solver setups (but, of course, *not* in differential analyzer setups utilizing high-gain amplifiers). Integrating capacitors have been indicated in dash lines in the computer setups of Fig. 3.19*b*.

The use of d-c integrators instead of high-gain d-c amplifiers often tends to eliminate troublesome high-frequency oscillations because of the intrinsically bad high-frequency response of integrators. The stability of the resulting computer setups must, however, still be investigated in each case.

The methods outlined above can also be applied to the solution of *sets of equations*, although the stability checks involved may become quite difficult.

The solution of equations, such as algebraic equations, which have several roots by means of ordinary equation solvers may be difficult or impractical.

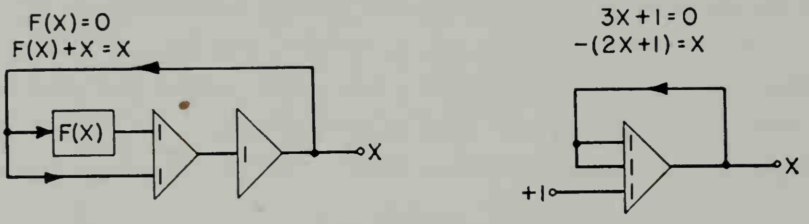

FIG. 3.19*a*. Solution of an equation by implicit computation. Stability tests are necessary in each case.

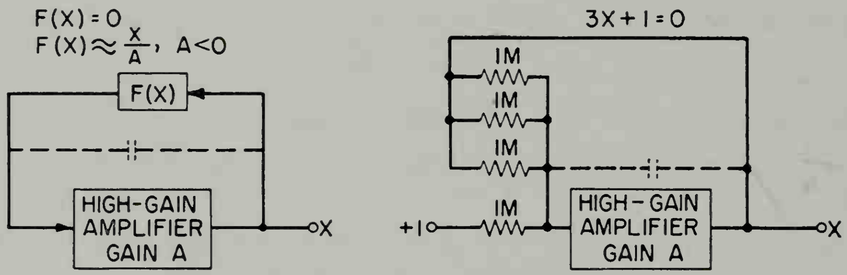

FIG. 3.19*b*. Use of high-gain amplifiers or d-c integrators (dash lines) for the solution of equations. Stability tests are necessary in each case.

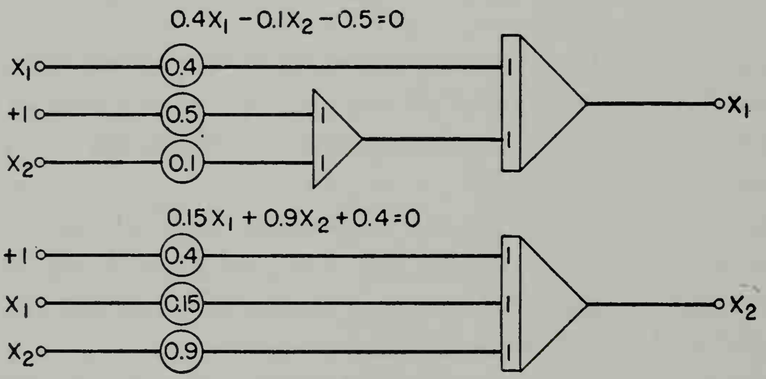

FIG. 3.19*c*. Solution of a set of linear simultaneous equations; d-c integrators are used as high-gain amplifiers.

The machine may "prefer" one of the real roots and solve for this root only. Again, an ordinary equation solver cannot solve directly for complex roots, and transformations introducing, for instance, the absolute values and the arguments of the complex roots as new real unknowns would have to be resorted to.

Solution of Sets of Linear Simultaneous Equations by Means of D-c Analog Computers. The use of d-c integrators instead of high-gain amplifiers for automatic equation solvers is particularly useful for the solution of sets of linear simultaneous equations by means of existing d-c analog-computer installations. The machine equations corresponding to a given set of linear simultaneous equations will have the form

$$\left.\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n + b_1 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n + b_n = 0 \end{array}\right\} \tag{3.33}$$

A computer setup for solving the above problem through a generalization of the scheme shown in Fig. 3.19*b* is obtained by setting up computing elements to establish the differential equations

$$\left.\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n + b_1 = -PX_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n + b_n = -PX_2 \end{array}\right\} \tag{3.34}$$

With properly designed integrators and reasonably low coefficients a_{ik}, the resulting computer setup will be stable if the real parts of all the roots of the characteristic equation

$$\begin{vmatrix} a_{11} - s & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - s & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - s \end{vmatrix} = 0 \tag{3.35}$$

are positive. If this is not the case, one may try to rearrange Eqs. (3.33) or to multiply some of the equations by -1. If the computer setup given by the machine equations (3.34) is stable, the machine variables X_1, $X_2, \ldots, X_n$ will approach the desired unknown values.

If stability cannot be obtained by the simple procedure described above, the unknowns $X_1, X_2, \ldots, X_n$ in (3.33) may be obtained by setting up a new set of machine equations

$$\left.\begin{array}{l} a'_{11}X_1 + a'_{12}X_2 + \cdots + a'_{1n}X_n + b'_1 = -PX_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a'_{n1}X_1 + a'_{n2}X_2 + \cdots + a'_{nn}X_n + b'_n = -PX_n \end{array}\right\} \tag{3.36}$$

with

$$a'_{ik} = \sum_{j=1}^{n} a_{ji}a_{jk}$$

$$b'_i = \sum_{j=1}^{n} a_{ji}b_j$$

$$i, k = 1, 2, \ldots, n$$

The new coefficients a'_{ik} will always lead to a characteristic equation (3.35) having positive real roots unless the determinant of the a_{ik} is equal to zero.

Figure 3.19*c* shows a computer setup for a simple numerical example. The unknown voltages X_1 and X_2 satisfying the set of equations

$$\left.\begin{aligned} 0.4X_1 - 0.1X_2 - 0.5 &= 0 \\ 0.15X_1 + 0.9X_2 + 0.4 &= 0 \end{aligned}\right\} \qquad (3.37)$$

are found as the "equilibrium voltages" approached by the solutions $X_1(\tau)$ and $X_2(\tau)$ of the set of differential equations

$$\left.\begin{aligned} 0.4X_1 - 0.1X_2 - 0.5 &= -PX_1 \\ 0.15X_1 + 0.9X_2 + 0.4 &= -PX_2 \end{aligned}\right\} \qquad (3.38)$$

after the initial transients have damped out. The solutions are stable, and in the equilibrium condition $PX_1 = PX_2 = 0$, so that the differential equations (3.38) reduce to Eqs. (3.37).

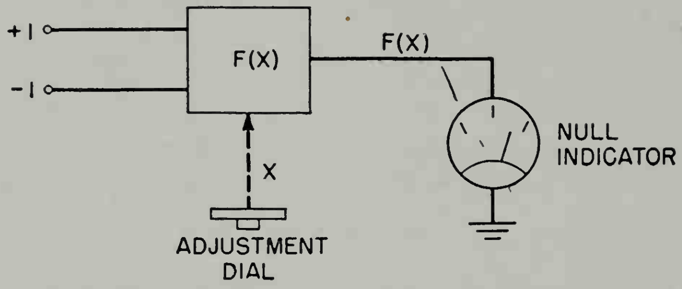

FIG. 3.20. Simple adjuster-type equation solver.

Adjuster-type Equation Solvers. The high computing speeds obtainable with d-c analog machines make it possible to solve equations with very simple equipment by a trial-and-error process. Figure 3.20 shows a computer setup forming a voltage $F(X)$ as a function of a dial rotation X by means of function potentiometers and summing networks. In order to solve the equation

$$F(X) = 0$$

for the unknown value or values of X, it is only necessary to adjust the dial until a meter or null indicator measuring the output voltage $F(X)$ reads zero. The value of the unknown may then be read on the adjustment dial. Devices of this type may be quite accurate (0.2 to 0.5 per cent), since the result is not affected by drift in the source voltage used.

In a more elegant version of the foregoing process, the voltage $F(X)$ may be computed as a function of an input voltage or shaft rotation X which is varied (swept) periodically through its range of variation as a known periodic function of the time. It is then possible to measure the time at which the null detector responds by automatic means, and the relative time at which this occurs during the sweep cycle may be used to measure each root of the given equation.

The principle of the adjuster-type computer may also be applied to the solution of sets of equations. Given a set of machine equations

$$\left.\begin{array}{l} f_1(X_1, X_2, \ldots, X_n) = 0 \\ f_2(X_1, X_2, \ldots, X_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ f_n(X_1, X_2, \ldots, X_n) = 0 \end{array}\right\} \tag{3.39}$$

where the f's are reasonable single-valued functions of the X's, perhaps the most generally applicable method of solution is to generate a voltage

$$F(X_1, X_2, \ldots, X_n) = f_1^2 + f_2^2 + \cdots + f_n^2 \tag{3.40}$$

as a function of the input voltages or shaft rotations $X_1, X_2, \ldots, X_n$ by means of suitable computing elements and to minimize this quantity by iterated adjustments of the X's.[1]

Other Applications of D-c Analog Computers. A d-c analog computer and its associated recording equipment may be very useful, quite apart from its more sophisticated applications, for *computing and plotting functions* of one or several variables. A d-c analog computer with a servo plotting table makes it easy to prepare reproducible graphs of functions such as τ^2, $\sin \tau$, e^τ, and many others. Again, complicated expressions involving transformations of empirically obtained functions may be computed and plotted if an input table or universal function generator (see Sec. 6.5) is available.

D-c analog computers are also very useful for performing simple integrations (definite and indefinite integrals). Such computations are involved, for instance, in the determination of weighted averages.

Integral Transforms. The computation of expressions of the type

$$F(x) = \int_a^b K(x,t)f(t)dt \tag{3.41}$$

where K and f are given functions of the variables x and t, is of the greatest importance in a large number of problems, as shown by the following examples: expansion of functions in series of orthogonal functions, Fourier analysis, convolution and superposition integrals, solution of integral equations, boundary value problems (Neumann and Dirichlet problems), computation of autocorrelation functions and cross-correlation functions.

Expressions of the form (3.41) frequently lend themselves conveniently to computation by means of d-c analog computers. MacNee[1] and Wallman[2] have made particularly interesting applications of repetitive computers to such computations. Some of these applications are so important that they warrant the design of special-purpose computers.

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[1] *Ibid.*

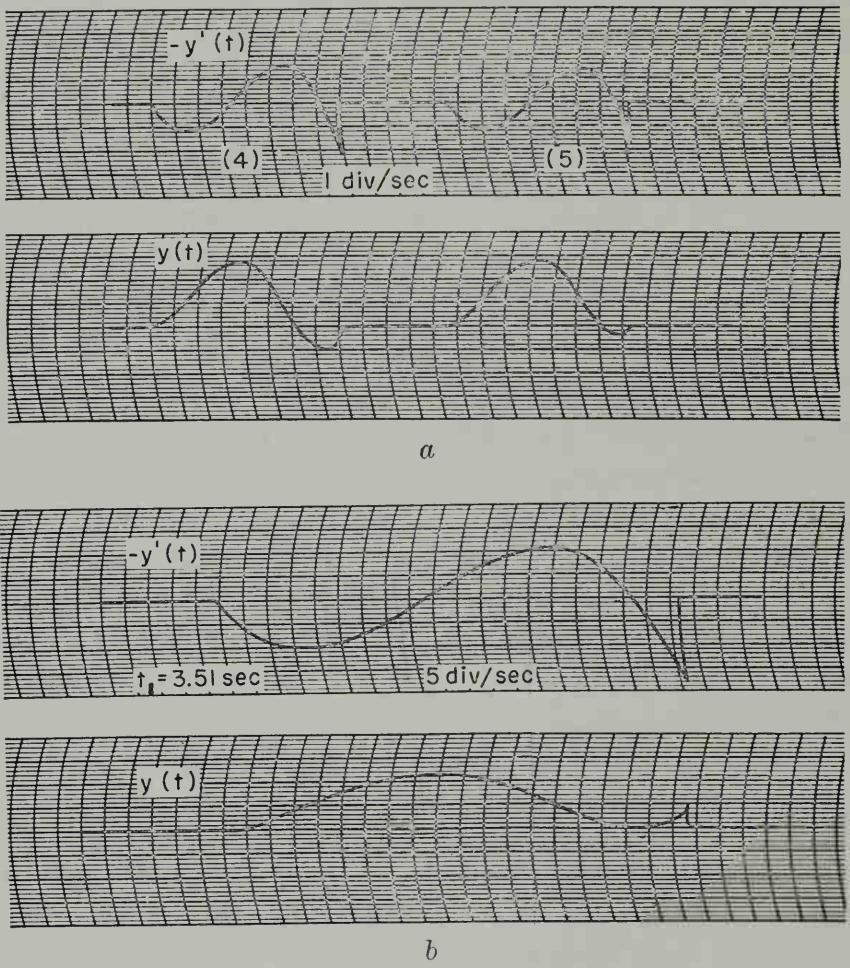

Fig. 3.21. Three steps of a trial-and-error process for the solution of the boundary value problem given by Eq. (3.42). The final solution, shown in (*b*), is presented with a faster recorder speed for greater accuracy (Hagelbarger, Howe, and Howe, *op. cit.*)

Solution of Boundary and Eigenvalue Problems by Means of D-c Analog Computers. In certain problems involving the solution of differential equations, the number of initial conditions (values of variables and derivatives for $t = 0$) given is not sufficient to determine the solution completely. Instead, the value of dependent variables or derivatives for some finite value t_1 are given. The solution of such *boundary value prob-*

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[1] MacNee, A. B., A High-speed Product Integrator, *MIT-RLE Rept.* 136, Aug. 17, 1949.

[2] Wallman, H., An Electronic Integral-Transform Computer and the Practical Solution of Integral Equations, *J. Franklin Inst.*, **250**: 45, 1950.

lems on a d-c analog computer proceeds by a trial-and-error process. The initial value settings left unspecified by the problem are varied until the solution satisfies the desired boundary conditions for $t = t_1$. Excellent examples of d-c analog-computer solutions of boundary value problems such as the static deflection of elastic beams are presented in the literature.[1] Figure 3.21, taken from the reference, shows several steps in the d-c analog-computer solution of a beam deflection problem involving the solution of a differential equation

$$p^4y = c \qquad \text{with } y]_0 = py]_0 = 0 \qquad \text{and} \qquad y]_{t=t_1} = py]_{t=t_1} = 0 \quad (3.42)$$

Many *eigenvalue problems*,[2] such as certain problems involving elastic vibrations or quantum mechanics, may be solved on d-c analog computers by trial-and-error procedures analogous to those used for the solution of ordinary boundary value problems. The values of the unknown eigenvalues must be varied until the desired boundary or normalization conditions are satisfied by the solutions of the given differential equations.[1] Frequently it is useful to obtain approximate solutions of eigenvalue problems on a d-c analog computer for use in powerful numerical methods involving successive approximations.[2]

Analog computers of the repetitive type (see Sec. 8.5) are particularly suitable for the rapid solution of boundary and eigenvalue problems, since such machines permit the effects of parameter variations to be observed directly on a cathode-ray oscillograph.

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[1] Hagelbarger, Howe, and Howe, *op. cit.*

[2] For a discussion of such problems, see H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry,* Van Nostrand, New York, 1946.

CHAPTER 4

THEORY AND DESIGN OF LINEAR COMPUTING ELEMENTS: OPERATIONAL AMPLIFIERS AND NETWORKS

4.1. Multiplication by Constant Coefficients: Potentiometers. *Introduction: Construction of Potentiometers.* A potentiometer consists of a resistance having terminals at each end and provided with a sliding contact or brush arranged so that it can be moved over the resistance. In the most common type of potentiometer, the resistance winding is arranged in cylindrical form while the brush is mounted on an arm which revolves on a central shaft.[1]

In d-c analog computers, potentiometers are most often used to multiply a machine variable by a constant coefficient a $(0 \geq a \geq 1)$ in the manner described in Sec. 1.3. This subject will be discussed in somewhat more detail here. The use of rotating motor-driven potentiometers for multiplication and function generation is described in Sec. 1.3 and in Chap. 6.

A potentiometer must be designed to certain specifications covering its *total resistance* and *power rating*. The resistances of potentiometers used in d-c analog computers vary between 10,000 and 100,000 ohms. The lower limit of the potentiometer resistance is determined by the power of the amplifier driving the potentiometer and by the required resolution (see page 117). The upper limit is fixed by loading considerations (see page 117) as well as by the availability of wire-wound resistance elements. Computing potentiometers should be capable of having a steady voltage of about 100 volts applied across a resistance of 10,000 ohms without appreciable heating; 4 watts is a commonly used power rating.

For accurate computation, the following properties are desirable in potentiometers used for setting constant coefficients:

1. *Mechanical ruggedness,* which will permit a potentiometer setting to be maintained in spite of vibrations, shock, etc.
2. *Freedom from microphonism,* or noise introduced by vibrations.
3. *Accuracy of calibration.* It should be possible to set the transfer function of the potentiometer to any desired value by means of a

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[1] Nettleton, L. A., and F. E. Dole, Potentiometers, *Rev. Sci. Instruments,* **17**: 356, 1946.

calibrated dial. If the dial scale is linear, the linearity of the potentiometer will be important.

4. *Resolution.* It should be possible to set the potentiometer to any desired coefficient. Accordingly, voltage "steps" due to movement of the sliding contact over the resistance element should be kept within a specified small value.
5. *Stability.* The resistance of the potentiometer, and especially the accuracy of the calibration, should not change appreciably with temperature and age.

The design and construction of potentiometers having such desirable properties have practically reached a state of perfection and have been described in great detail by F. Dole.[1]

Most of the above requirements will be satisfied only by potentiometers using wire-wound resistance elements, although some types of resistance elements consisting of metallized ceramic materials may be satisfactory.

The potentiometers used most frequently for coefficient settings are the helical card types (*Gibbs Micropots* and *Beckman Helipots*). The entire scale of such a potentiometer is covered in 5 to 20 turns of the specially constructed dial. The latter can be read to 1 part in 10,000, which corresponds to the 0.1 per cent linearity and resolution of such potentiometers.

Corrections for Potentiometer Loading. Ordinarily, the dial reading a of a computing potentiometer refers to the fraction ar of the total potentiometer resistance r across which the output voltage X_o is measured. If such a potentiometer is connected to a load of infinite impedance (see Fig. 4.1a), the potentiometer transfer function will be

$$\frac{X_o}{X_1} = \frac{ar}{r} = a \tag{4.1}$$

which corresponds to the dial reading.

If a load of resistance r_L is connected to the potentiometer output terminals (see Fig. 4.1b), Eq. (4.1) holds no longer. From the nodal equation for the output terminal,

$$(X_o - X_1)\frac{1}{(1-a)r} + \frac{X_o}{ar} + \frac{X_o}{r_L} = 0 \tag{4.2}$$

it follows that

$$\frac{X_o}{X_1} = \frac{a}{1 + a(1-a)\dfrac{r}{r_L}} = a - \frac{a^2(1-a)\dfrac{r}{r_L}}{1 + a(1-a)\dfrac{r}{r_L}} \tag{4.3}$$

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[1] Blackburn, J. F., *Components Handbook*, MIT Radiation Laboratory Series, Vol. 17, McGraw-Hill, New York, 1949, Chap. 8.

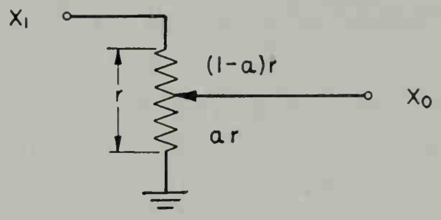

FIG. 4.1*a*. Coefficient setting potentiometer.

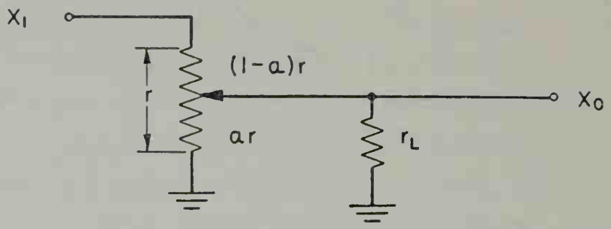

FIG. 4.1*b*. Coefficient setting potentiometer with load.

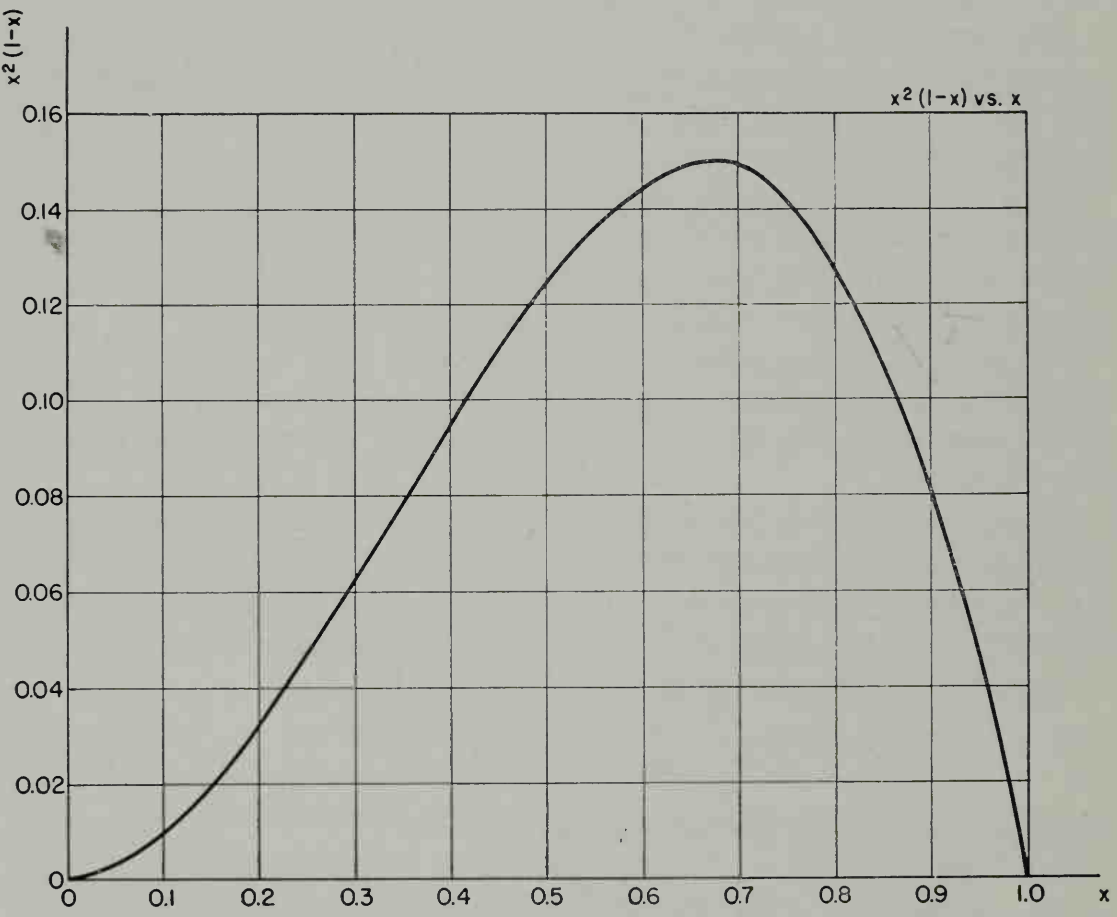

FIG. 4.1*c*. Potentiometer-loading correction chart based on the relation

$$a - \frac{X_o}{X_1} \approx a^2(1 - a)\frac{r}{r_L}$$

Multiply the ordinates corresponding to a desired value of $x = \frac{X_o}{X_1}$ by the value of $\frac{r}{r_L} \leq 0.3$; this quantity must be added to the desired value of the transfer function $\frac{X_o}{X_1}$ to obtain the correct potentiometer setting a. The use of charts of this type is not recommended if setting accuracies better than 0.1 per cent are desired, and tables based on the exact formula (4.4) should be used.

This is the correct relation between the potentiometer setting a and the transfer function X_o/X_1. The quantity

$$a - \frac{X_o}{X_1} = \frac{a^2(1 - a)\dfrac{r}{r_L}}{1 + a(1 - a)\dfrac{r}{r_L}} \qquad \left[= a^2(1 - a)\frac{r}{r_L} \text{ for } \frac{r}{r_L} \ll 1 \right] \tag{4.4}$$

is a measure for the error made by applying the simplified relation (4.1) instead of the correct relation (4.3).

Tables or graphs based on the relation (4.3) between a and X_o/X_1 (see Fig. 4.1*c*) make it relatively easy to introduce the proper *loading corrections* in the potentiometer dial settings.

The most common load for a computing potentiometer is an input resistor R_i of a summing amplifier, phase inverter, or parallel-feedback integrator. Since the grid of a high-gain operational amplifier is practically at ground potential, one has, in all such cases,

$$r_L = R_i \tag{4.5}$$

Example. It is desired to use a 20,000-ohm potentiometer to multiply an input voltage by 0.4. The only potentiometer load is the 1-megohm input resistor of an integrator. Accordingly, $r_L/r = 50$; reference to Fig. 4.1*c* shows that the correct dial setting is

$$a = 0.4 + 0.002 = 0.402$$

which may be verified by Eq. (4.3). If the loading correction is neglected, an error of 0.5 per cent results.

Errors due to loading may be serious in the case of motor-driven variable coefficient potentiometers, since it is difficult to apply loading corrections in such cases. Special circuit arrangements for the reduction of loading errors are discussed in Chap. 6.

External Calibration. The cost of a number of high-precision linear potentiometers together with their associated dials may account for an appreciable part of the total computer cost. A considerable saving can sometimes be effected by using inexpensive, rugged, wire-wound radio volume controls for coefficient settings. Such potentiometers may be set accurately to any desired coefficient by comparison of their setting with that of a precision-calibrated potentiometer. Figure 4.2 shows the potentiometer to be set connected across the same test voltage source (direct current or low-frequency alternating current) as the comparison potentiometer. The latter is first adjusted to the desired setting. The low-cost potentiometer is then adjusted until its output voltage equals that of the comparison potentiometer, as indicated by a null on the galvanometer, oscilloscope, etc., used as a null detector. *Since this setting* *procedure may be performed with the potentiometer load (or an equivalent resistance) connected (see Fig. 4.2), no loading corrections are necessary.* The comparison circuit, which incidentally is capable of performing many other useful functions (see, for instance, Secs. 6.6 and 7.7), should be arranged so that it can be instantly connected in turn to any of the potentiometers in the computer proper. This may be accomplished conveniently by means of jacks or push-button switches associated with each potentiometer.

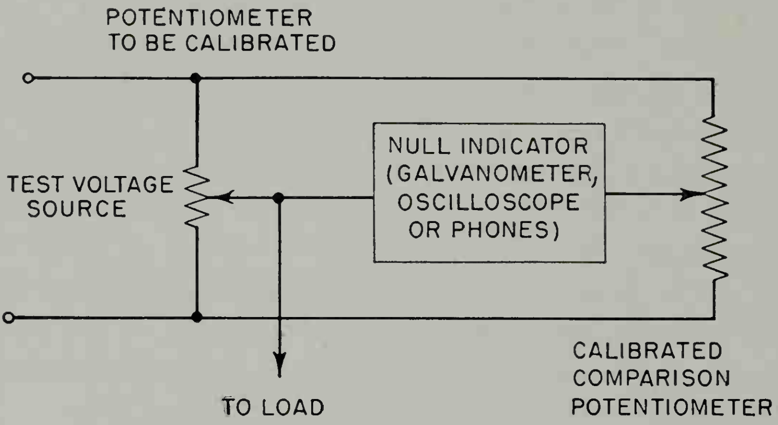
POTENTIOMETER
TO BE CALIBRATED
TEST VOLTAGE SOURCE
NULL INDICATOR (GALVANOMETER, OSCILLOSCOPE OR PHONES)
TO LOAD
CALIBRATED COMPARISON POTENTIOMETER

FIG. 4.2. Arrangement for setting or calibrating potentiometers by means of a calibrated comparison potentiometer.

Fine Adjustments. If the resolution of a potentiometer (especially one of low cost) is found to be insufficient, a *vernier adjustment* can be provided by means of a second potentiometer or variable resistor of smaller resistance as shown in Fig. 4.3.

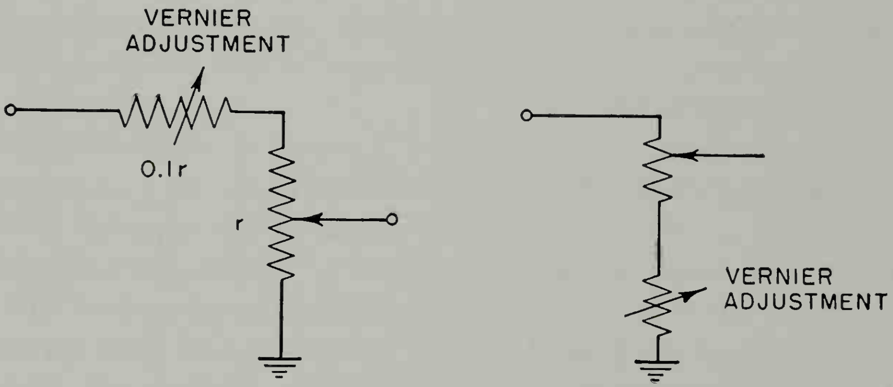

FIG. 4.3. Circuits permitting fine adjustment of potentiometer settings.

Other Schemes for Multiplying by Constant Coefficients. In some cases, particularly in low-cost computers, it is desirable to avoid the use of the power amplifiers necessary to drive low-impedance computing potentiometers. In these instances, it is possible to perform multiplications by constant coefficients by varying impedances (usually resistors) in the operational amplifiers. Figure 1.9 shows how such multiplications can be realized through the use of combinations of resistors in a summing amplifier. In the Goodyear computer (see Sec. 8.6), all summing resistors are removable and may be chosen in accordance with the desired coefficients. Figure 4.4 shows operational amplifiers permitting continuous adjustment of coefficients.

Sometimes it is possible to replace potentiometers by special lattice networks having a high input impedance as well as the same output impedance for all coefficient settings. Because of the difficulty of ganging variable resistors accurately, such arrangements are not too suitable if continuously variable coefficients are desired.

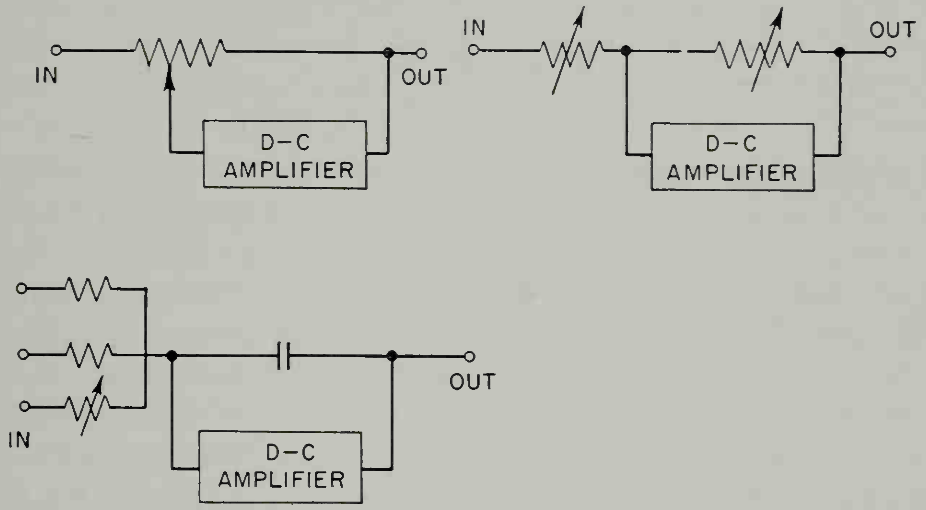

FIG. 4.4. Circuits permitting the introduction of adjustable constant coefficients without the use of low-impedance potentiometers. No loading difficulties are experienced with these circuits, and the use of power amplifiers for driving potentiometers can be eliminated.

4.2. The Design of Summing Amplifiers: Calibration and Error Analysis. *Introduction: Performance Equation.* The type of summing amplifier commonly employed in d-c analog computers was described in Sec. 1.3, where its performance equation (1.17) was also derived. The circuit of this summing amplifier is shown in block-diagram form in Fig. 1.7*d*; the performance equation was found in Sec. 1.3 to be

$$X_0 = \left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right) \frac{AR_o}{(1 - A) + R_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}\right)}$$

$$= -R_o \sum_{i=1}^{n} \frac{X_i}{R_i} \frac{(-A)}{(1 - A) + R_o \sum_{i=1}^{n} \frac{1}{R_i}} \qquad (4.6)$$

where the forward gain A of the d-c amplifier must be negative in order to keep the amplifier stable. In high-gain amplifiers with

$$|A| \gg 1 \qquad \text{and} \qquad |A| \gg R_o \sum_{i=1}^{n} \frac{1}{R_i}$$

the output voltage is practically independent of the gain, and Eq. (4.6) reduces to

$$X_o = -R_o \sum_{i=1}^{n} \frac{1}{R_i} \tag{4.7}$$

The negative sign shows that the summing amplifier shown in Fig. 1.7*d* will act as a *phase inverter*, which is an important function in itself.

Calibration Accuracy. With the high values (over 1,000) of the forward gain A encountered in most practical computing amplifiers, the simplified performance equation (4.7) offers a convenient means of calibrating summing amplifiers. Instead of making voltage measurements to determine the amplifier output voltage X_o as a function of the input voltages $X_1, X_2, \ldots, X_n$, it is permissible to use precision resistors (accurate to at least 0.1 per cent for 0.1 per cent calibration accuracy) for $R_o, R_1, R_2, \ldots, R_n$ and to substitute their values into Eq. (4.7).

If the latter equation is to be used intelligently in this manner, however, it is necessary to estimate the magnitude of the error due to substituting the simplified equation (4.7) for the performance equation (4.6). The fractional error will, in each case, be equal to the difference between the values of X_o given by Eqs. (4.6) and (4.7) divided by the value of X_o given by Eq. (4.6). It follows that the percentage error ϵ is given by

$$\epsilon = \frac{100}{A}\left(1 + R_o \sum_{i=1}^{n} \frac{1}{R_i}\right) \tag{4.8}$$

If, then, the percentage error is not to exceed a given value ϵ, one must have[1]

$$A \geq \frac{100}{\epsilon}\left(1 + R_o \sum_{i=1}^{n} \frac{1}{R_i}\right) \tag{4.9}$$

which determines the minimum gain for given calibration accuracy.

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[1] The condition (4.9) is somewhat more accurate than the similar condition given in G. A. Korn, Introduction to D-c Analog Computers, *Electronics*, April, 1948. [No approximations are involved in (4.9).]

Effects of Gain Changes and Distortion. The inverse feedback inherent in the summing amplifier circuit helps to reduce the effects of gain changes and distortion in the d-c amplifier on the summing amplifier output voltage.

As a matter of fact, it is clear that, if the condition (4.9) is satisfied throughout the range of possible gain changes, the effect of these gain changes on the summing amplifier calibration will not exceed the specified error limits.

Amplitude distortion of ϵ_D per cent may be considered as a gain change $(\epsilon_D/100)A$ applicable to only certain portions of the signal. The effects of distortion in summing amplifiers are thus reduced in the same manner as the effects of amplifier gain changes. In order to keep the value of ϵ_D reasonably small, care should be taken to keep the amplifier output voltage and the load admittance below specified design limits (see also Sec. 5.2).

High-frequency Response Requirements. Since the gain A of any amplifier is really not constant but must fall off at high signal frequencies, it is important to know how this loss of gain will affect the performance of a summing amplifier. For accuracy in computation, the following conditions must be satisfied:

1. The absolute value of the amplifier forward gain A must remain above the limit set by (4.9) for all anticipated signal frequencies.
2. The manner in which the forward gain $A(P)$ depends on the frequency (this includes the phase shift involved) must be such that the summing amplifier is *stable.* This will be the case if the roots of the equation

$$[1 - A(P)] + R_o \sum_{i=1}^{n} \frac{1}{R_i} = 0$$

   [obtained by equating the denominator of (4.6) to zero] all have negative real parts.[1]

Finally, the phase shift between any input voltage X_i of a summing amplifier and the output voltage X_o should be small throughout the range of signal frequencies (less than 0.1 deg by rule of thumb) if no integrating or differentiating action is to take place. Such phase shifts tend to increase with the ratio R_o/R_i. Again, cascading of computing amplifiers tends to increase both gain and phase shifts, so that attempts to multiply machine variables by factors in excess of 100 sometimes result in instability.

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[1] Bode, H. W., *Network Analysis and Feedback Amplifiers,* Van Nostrand, New York, 1945.

Effect of Source and Load Impedance. In order to assess the performance of the summing amplifier in a computing system, it also may be necessary to determine the effect of the source and load impedances to which the amplifier is connected upon the performance of the amplifier. This effect will be considered in the Appendix.

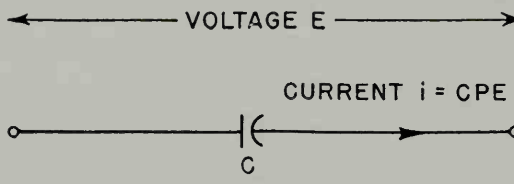

FIG. 4.5. Relation between voltage and current in a capacitive circuit.

4.3. Integrator Performance.

Introduction. It was pointed out in Sec. 1.4 that the design of the integrators is often a crucial part of the computer design. Integrators were seen to be a convenient means for introducing relations between *changes* of machine variables in d-c analog computers with the time τ usually serving as the independent variable.

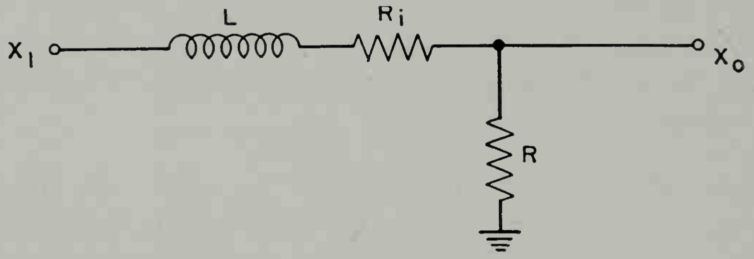

FIG. 4.6. An integrating network comprising an inductance.

The physical basis of most electrical integrators is the basic property of capacitors[1] illustrated in Fig. 4.5. Neglecting the effect of capacitor leakage, the current i through the capacitor is always proportional to the time derivative of the voltage across the capacitor:

$$i = CPE \tag{4.11}$$

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[1] It is at least theoretically possible to design integrators using networks comprising a resistance and an inductance. An integrating network using such elements is shown in Fig. 4.6. The transfer function of this circuit is

$$\frac{X_o}{X_1} = \frac{R}{R + R_i + LP} = \frac{R}{R + R_i} \, \frac{1}{\left(\dfrac{L}{R + R_i}\right) P + 1} \tag{4.10}$$

where R_i is the inherent resistance of the inductor. At the present time, circuits of this type are used less frequently than RC-type integrating circuits because the required inductances may be as large as 20 to 50 henrys and thus costly and physically large. Furthermore, such inductances tend to have an appreciable inherent resistance R_i and usually need iron cores which may become saturated and thus make the circuit constants dependent on the input voltages.

In this chapter, the basic integrating network of Fig. 1.7*e* will be discussed in more detail. In addition, other integrating devices, which constitute attempts to improve on the original circuit, will be presented.

Simple RC Integrating Network. In the simple RC integrating network shown in Fig. 4.7, an input voltage X_1 will begin to charge the capacitor C through the resistor R in a manner approximating true integration. The output voltage X_o built up on C during this process will, however, tend to oppose the input charging voltage and may thus make accurate integration impossible. The situation is further complicated through the existence of the leakage resistance R_L which is due to capacitor leakage in addition to any load which may be connected to the integrator output terminals.

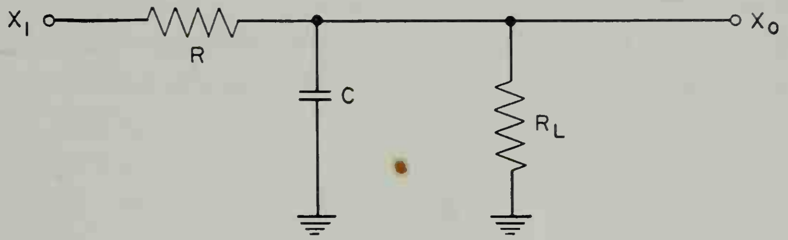

FIG. 4.7. Simple integrating network with load.

The nodal equation of this network leads to the *transfer function* X_o/X_1 of the integrating network

$$\frac{X_o}{X_1} = \frac{R_L}{R + R_L} \frac{1}{\dfrac{R_L}{R + R_L} RCP + 1} \tag{4.12}$$

If the leakage resistance R_L is infinitely large, Eq. (4.12) reduces to the simplified equation (1.23) given in Chap. 1.

General Integrator Performance Analysis: Time Constant and Attenuation. It should be mentioned at this point that *the performance of most d-c integrators may be described in terms of transfer functions of the form*

$$\frac{X_o}{X_1} = \frac{k}{bP + 1} \tag{4.13}$$

of which Eq. (4.12) is a special case. The coefficient b of P in the denominator is called the *time constant* of the integrator and will be seen to constitute a measure of the integrator accuracy. The parameter k/b corresponds to the *gain* (or rate gain) of the integrator. The practical meaning of integrator time constant and gain will be made clearer by the following analysis.

Output Error and Rate Error of an Integrator. The *output error* of an integrator will be the difference between the integrator output voltage X_o

and the output voltage $(k/bP)X_1$ of a "perfect" integrator of equal gain. For an integrator described by the performance equation (4.13), the output error $e(\tau)$ will be

$$\begin{aligned} e(\tau) &= X_o - \frac{k}{b} X_1 = \left[\frac{k}{bP+1} - \frac{k}{bP}\right] X_1 \\ &= -\frac{k}{bP(bP+1)} X_1 = -\frac{1}{bP} X_o \end{aligned} \tag{4.14}$$

for any value of the time τ. It must be remembered that the "linear" output error (4.14) will in general have to be added to other errors due to calibration changes, distortion, and drift.

It is clear that the derivative of the output voltage of any d-c integrator should be as nearly as possible proportional to the input voltage. In fact, for a perfect integrator one would have

$$PX_o = \frac{k}{b} X_1 \tag{4.15}$$

The amount $Pe(\tau)$ by which the time derivative PX_o of the output voltage X_o varies from the value given by Eq. (4.15) shall be called the *rate error* of the integrator under consideration and constitutes a useful measure of the integrator accuracy. As a matter of interest, an integrator whose rate error is always zero will have only errors which can be compensated for by simply changing the initial conditions of the problem.

For an integrator having the performance equation (4.13), the derivative of the output voltage X_o will be

$$PX_o = \frac{kP}{bP+1} X_1 \tag{4.16}$$

Assuming that the entire error in the derivative PX_o is due to the difference between Eqs. (4.15) and (4.16), the rate error Pe of the integrator is then

$$Pe(\tau) = PX_o - \frac{k}{b} X_1 = -\frac{1}{b}\frac{1}{bP+1} X_1 = -\frac{1}{b} X_o \tag{4.17}$$

Note again that the effects of drift, distortion, and calibration changes have not been considered here. It is seen that both $e(\tau)$ and $Pe(\tau)$ will decrease as the integrator time constant b is increased.

Step-function Response of a D-c Integrator: Effect of Computing Time on Error. If an input voltage step function $c1(\tau)$* (*e.g.*, from a constant-voltage source suddenly switched into the circuit) is applied to an integra-

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* The definition of the unit step function $1(\tau)$ is given by

$$1(\tau) = 0 \text{ for } \tau < 0 \qquad 1(\tau) = 1 \text{ for } \tau \geq 0$$

tor described by a transfer function of the form (4.13), the circuit response will be

$$X_o(\tau) = kc\,(1 - e^{-\frac{\tau}{b}}) = c\left[\left(\frac{k\tau}{b}\right) - k\left(\frac{\tau^2}{2b^2} \pm \cdots\right)\right], \tau \geq 0 \quad (4.18)$$

This is easily verified by substitution into the differential equation (4.13). It is seen that the output voltage X_o begins to change linearly with time at a rate equal to $(k/b)c$. This would correspond to correct integration of the input step function. With increasing computing time τ, however, the output voltage X_o no longer changes linearly but tends to approach a limiting value[1] kc exponentially, as shown by Eq. (4.18) and Fig. 4.8. (As in the case of the simple RC integrator, this is usually due to a "counter-emf" developed as a result of the charge on an integrating capacitor.)

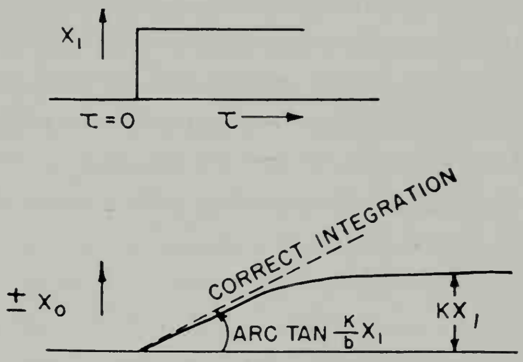

Fig. 4.8. Integration of a step function.

The resulting (linear) output error e (which is in addition to errors due to distortion, drift, etc., in the amplifiers involved) is given by the second term on the right of Eq. (4.18). The integration error is seen to increase with time; if $1 > \tau/b \geq 0$, the absolute value of the error e will be less than, or equal to, the first term of the error series, or

$$|e| \leq \frac{\tau^2}{2b^2} kc \quad (4.19)$$

For the *percentage error* ϵ, it follows that

$$\epsilon \leq 50 \frac{\tau}{b} \quad (4.20)$$

Hence, *if the resultant output voltage is to be within* ϵ *per cent of the true integral* $(k/b)c\tau$, *the computing time* τ *must be limited so that*

$$\tau \leq \frac{b\epsilon}{50} \quad (4.21)$$

The *rate error Pe* for step-function input is obtained by substituting Eq. (4.18) into (4.17):

$$-Pe(\tau) = \frac{k}{b} c\,(1 - e^{-\frac{\tau}{b}}) \quad (4.22)$$

---

[1] In actual practice, this limiting voltage kc may be higher than the overload or saturation voltage of the computing elements in question, so that the output voltage X_o cannot reach the limiting value kc.

It is seen that

$$|Pe(\tau)| \leq \frac{k}{b^2} c\tau \qquad \left(1 > \frac{\tau}{b} \geq 0\right) \tag{4.23}$$

If the integrator comprises only linear networks and amplifiers, *the relations* (4.20) *and* (4.23) *are not restricted to step-function input voltages,* since the operation of a linear network on *any* input voltage may be considered in terms of a superposition of step-function voltages, with all steps but the first one being integrated for time shorter than τ.

It should be remembered that the relations (4.19) to (4.23) refer only to errors due to d-c integrators as such. The effects of such errors on the over-all accuracy of a differential analyzer must be discussed separately.

Steady-state Sinusoidal Response. It is often important to know the response of an integrator to steady-state sinusoidal voltages, in particular since, again, any input voltage may be represented as a superposition of such voltages. In the steady state, both X_1 and X_o will be sinusoidal voltages represented by[1]

$$\tilde{X}_1 = \bar{X}_1 e^{j\omega\tau} \qquad \tilde{X}_o = \bar{X}_0 e^{j\omega\tau} \tag{4.24}$$

where ω is the circular frequency. Substitution of (4.24) into the differential equation (4.13) shows that

$$\bar{X}_o = \frac{k}{bj\omega + 1} \bar{X}_1 = \frac{k\bar{X}_1}{b\omega j} - \frac{1}{b\omega j} \frac{k}{bj\omega + 1} \bar{X}_1 \tag{4.25}$$

where the first term on the right corresponds to true integration and the second term measures the absolute value and phase of the (complex) integration error $\bar{e}$ for sinusoidal input voltages of circular frequency ω. The *absolute value* of the error is

$$|\bar{e}| = \left|\frac{\bar{X}_1}{b\omega} \frac{k}{\sqrt{1 + (b\omega)^2}}\right| \leq \left|\frac{\bar{X}_1 k}{b^2\omega^2}\right| \tag{4.26}$$

whereas the *phase error* [the amount by which the angle $\sphericalangle\ (\bar{X}_o, \bar{X}_1)$ differs from 90 deg] δ is given by

$$\tan \delta = \frac{1}{b\omega} \tag{4.27}$$

Equations (4.25) to (4.27) further illustrate the importance of the parameters b and k, since any input voltage may be thought of as made up of sinusoidal components. It is seen from Eqs. (4.26) and (4.27) that high-frequency components of any input voltage will be integrated more accurately than low-frequency components.

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[1] Refer to W. R. Symthe, *Static and Dynamic Electricity,* McGraw-Hill New York, 1939, on the method of using complex numbers to describe the amplitude and phase characteristics of alternating currents and voltages.

Time Constant and Gain of the Simple RC Integrator. According to (4.12), the time constant b of the simple RC integrating network shown in Fig. 4.7 is

$$b = \frac{R_L}{R + R_L} RC \tag{4.28}$$

and the gain k/b is

$$\frac{k}{b} = \frac{1}{RC} \tag{4.29}$$

If the effect of leakage and loading is negligible ($R_L \to \infty$), Eqs. (4.28) and (4.29) reduce to

$$b = RC \qquad \frac{k}{b} = \frac{1}{RC} \tag{4.30}$$

It may be concluded that

1. In the simple RC network, large values of the time constant b desirable for good integration can be obtained only at the expense of large values of R and C and of low gain.
2. The effect of connecting a load to a simple RC integrator is to decrease the time constant b and thus the accuracy of integration.

The RC network has, on the other hand, the obvious advantages of being simple, inexpensive, and free of drift, since it requires no vacuum tubes. RC integrators will often be satisfactory in cases involving relatively short computing times or for input voltages containing only relatively high-frequency components.

Initial Conditions. It was mentioned already in Chaps. 1 and 2 that each integrator must be provided with means for introducing initial condition voltages into the computer setup. Practical circuits for accomplishing this purpose are described in Sec. 7.1.

4.4. Some Electronic Integrators. *Introduction: The Perfect Integrator.* The electronic integrators to be discussed in this and the following sections constitute attempts to improve the performance of the basic RC network, essentially by reducing the effects of the "counter-emf" which tends to oppose the charging current of an integrating capacitor. Such electronic circuits will, then, permit higher values of the integrator time constant without affecting the integrator gain detrimentally.

McCool[1] has pointed out that the problem of integrator design may be reduced to that of developing a perfect current generator (with zero shunt admittance) whose current output is accurately proportional to the integrator input voltage X_1. The current integrator is then used to charge an integrating capacitor (Fig. 4.9); according to (4.11), the voltage on the capacitor will be the time integral of the current and thus of the input voltage X_1. The reader will find it instructive to interpret the electronic integrators in this chapter as attempts to realize such an arrangement.

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[1] McCool, W. A., An Improved Analog Computing Circuit, *NRL Rept.* P-3423 (unclassified), Feb. 23, 1949.

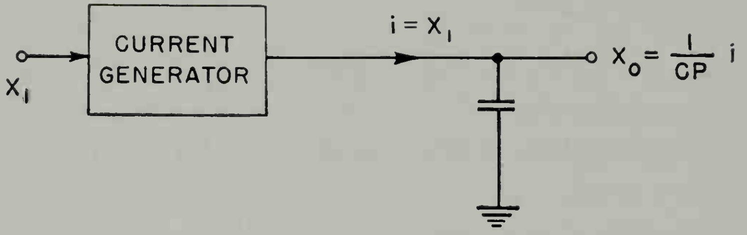

FIG. 4.9. Equivalent circuit of a perfect integrator.

Looking at the question of a "perfect" integrator[1] from the most general possible viewpoint, it can be said that the physical realization of a true integrator with a transfer function

$$\frac{X_o}{X_1} = \frac{k}{b}\frac{1}{P} \tag{4.32}$$

is theoretically possible and may be approached in practice by increasing both b and k in transfer functions such as (4.13). Since the ideal transfer function (4.32) implies the use of a device having infinite gain at zero frequency, a true integrator must involve an "active" circuit with critical regeneration; no passive network can satisfy this requirement. Of the electronic integrators discussed in this chapter, the bootstrap integrator (see page 137) and the regenerative feedback integrator (see Sec. 4.7) do make use of regenerative amplifiers and are theoretically capable of integrating perfectly.

In general, practical electronic integrating devices are capable of *approximating* the ideal transfer function (4.32), throughout a useful range of frequencies, with or without the use of regenerative amplifiers. In each case, a careful *accuracy analysis* will be necessary to determine the

---

[1] It should be mentioned in passing that, unlike the true integrator, the "perfect" differentiator is not physically realizable. The ideal differentiator would have a transfer function

$$\frac{X_o}{X_1} = \pm P \tag{4.31}$$

This would imply infinite gain at infinite frequency, a condition impossible to attain in practice. Reasonable approximations to (4.31) may be obtained for limited frequency ranges. A number of circuits suitable for electronic differentiation are presented in the Appendix.

errors involved in the approximation. Such an accuracy analysis is presented in Sec. 4.5 for a frequently used type of integrator.

The question of setting up *initial conditions* in integrators will be taken up in Sec. 7.1.

RC Integrating Network with Buffer Amplifier. The disadvantages which result from loading an *RC* integrating network may be largely overcome through the use of a buffer amplifier, as indicated in Fig. 4.10. The buffer amplifier (which might be a cathode follower or have a cathode-follower input stage) presents a high impedance to the integrating network and isolates it from the load. The amplifier may also be used to increase the integrator gain, but high values of R and C may still be needed to obtain a satisfactory time constant RC. The use of a buffer amplifier cannot, of course, overcome the loading effect of the capacitor leakage resistance. Finally, unless special precautions are taken (see Chap. 5), the grid current of the buffer amplifier may present a further problem by acting as a spurious input source which tends to charge the capacitor and thus to cause drift in the integrator output.

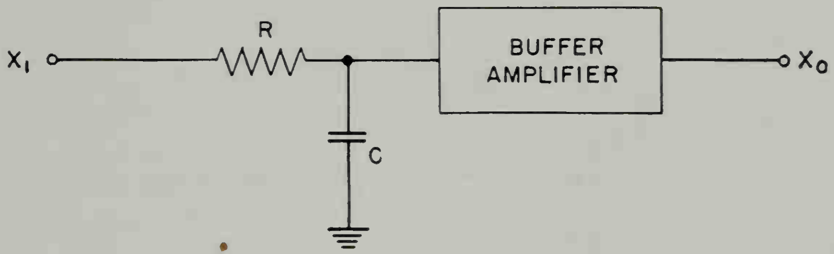

FIG. 4.10. Simple integrating network with buffer amplifier.

RC Integrators Using "Amplified Resistance." Another class of electronic integrators attempts to increase the integrating resistance R of an RC network by replacing the resistor R in Fig. 4.7 with the "dynamic plate resistance" of a vacuum tube. Figure 4.11*a* shows a vacuum-tube circuit with current feedback introduced by means of the cathode resistor R_K. In this circuit, the tube acts like a voltage source tending to oppose any current changes Δi in the cathode resistor, as indicated by the equivalent circuit[1] shown in Fig. 4.11*b*, where μ and r_p are the amplification factor and plate resistance of the tube, respectively. As a result, the tube looks like a resistance ("dynamic plate resistance")

$$Z = [r_p + (1 + \mu)R_K] \tag{4.33}$$

to any reasonably small voltage change applied at the terminals shown (Fig. 4.11*c*), while the steady-state d-c voltage across the tube and its

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[1] For a discussion of vacuum-tube equivalent circuits, see D. G. Fink, *Engineering Electronics*, McGraw-Hill, New York, 1938.

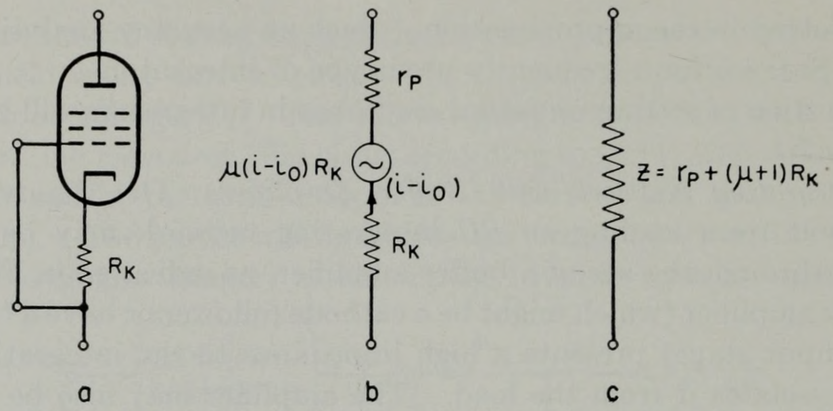

Fig. 4.11. Equivalent circuits illustrating the dynamic plate resistance of a vacuum tube. Because of the current feedback, the circuit tends to oppose changes $(i - i_o)$ in the plate current.

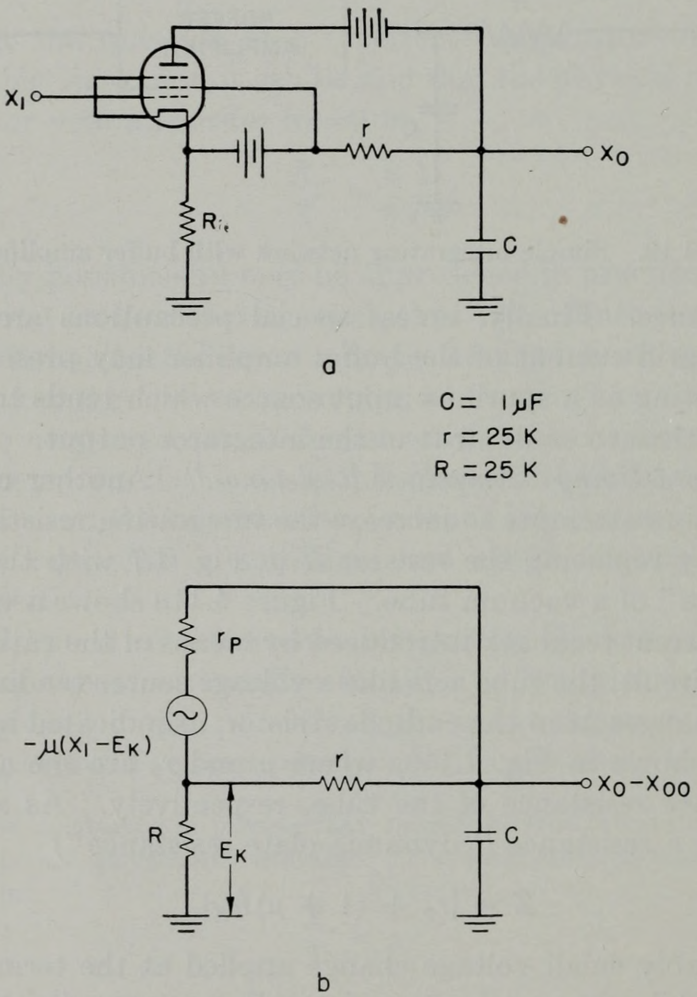

Fig. 4.12. Electronic integrator based on resistance amplification and equivalent circuit.

cathode resistor still has a reasonable value. It may be said that the tube has "amplified" the resistance R_K of the cathode resistor. Such resistance amplification is particularly effective with pentodes because of the high effective value of μ.

The application of this principle requires considerable ingenuity, since the tube supply voltages must not be permitted to interfere with the operation of the integrating circuits. In the integrator circuit shown in Fig. 4.12*a*,[1] use is made of the "dynamic plate resistance" of a tube (triode or pentode) to obtain an "amplified time constant." The *equivalent circuit for voltage changes* is shown in Fig. 4.12*b*. Analysis of this circuit yields the transfer function for voltage changes or *dynamic transfer function*

$$\frac{X_o - X_{o0}}{X_1} = \frac{-\mu}{\dfrac{r_p}{r} + 1} \, \frac{1}{\dfrac{\dfrac{r_p}{R} + (\mu + 1) + \dfrac{r_p}{r}}{\dfrac{r_p}{r} + 1} RCP + 1} \tag{4.34}$$

In this analysis, X_{o0} is the value assumed by X_o for $X_1 = 0$; X_{o0} may be referred to as the quiescent, zero signal, or reference output voltage with respect to ground. The time constant b and the gain k/b of this integrator are, respectively,

$$b = \frac{\dfrac{r_p}{R} + (\mu + 1) + \dfrac{r_p}{r}}{\dfrac{r_p}{r} + 1} RC \qquad \frac{k}{b} = \frac{-\mu}{RC\left[\dfrac{r_p}{R} + (\mu + 1) + \dfrac{r_p}{r}\right]} \tag{4.35}$$

In the circuit of Fig. 4.12*a*, the resistor r shunting the vacuum tube is necessary to provide a path for the d-c plate supply voltage. In order to decrease this shunting effect while still maintaining the d-c plate voltage, McCool[2] has ingeniously replaced r by the dynamic plate resistance Z of a second tube, as shown in Fig. 4.13. The dynamic transfer function of the improved circuit is

$$\frac{X_o - X_{o0}}{X_1} = \frac{-\mu}{\dfrac{r_p}{Z} + 1} \, \frac{1}{\dfrac{\dfrac{r_p}{R} + (\mu + 1) + \dfrac{r_p}{Z}}{\dfrac{r_p}{Z} + 1} RCP + 1} \tag{4.36}$$

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[1] This circuit seems to have been described in the literature first by G. A. Philbrick, Designing Industrial Controllers by Analog, *Electronics*, June, 1948.

[2] McCool, *op. cit.*

with

$$b = \frac{\dfrac{r_p}{R} + (\mu + 1) + \dfrac{r_p}{Z}}{\dfrac{r_p}{Z} + 1} RC \quad \text{and} \quad \frac{k}{b} = \frac{-\mu}{RC\left[\dfrac{r_p}{R} + (\mu + 1) + \dfrac{r_p}{Z}\right]} \tag{4.37}$$

In the integrator circuit of Fig. 4.14*a*, one tube, V_1, is used to "amplify" its cathode resistance R_K, whereas another similar tube, V_2, acts as a voltage amplifier and increases the gain of the circuit. The *equivalent circuit for voltage changes* is shown in Fig. 4.14*b*, and the nodal equation is

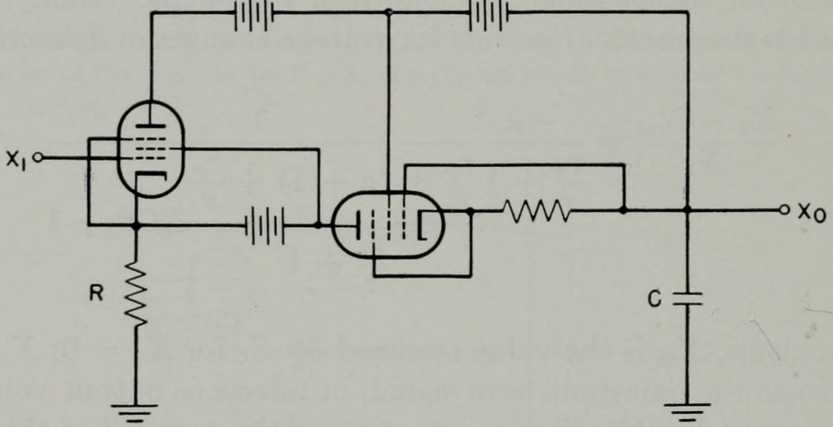

FIG. 4.13. Improved electronic integrator based on resistance amplification.

$$0 = (X_o - X_{o0})\left(\frac{1}{R_L} + \frac{1}{Z} + CP\right) + [(X_o - X_{o0}) + \mu X_1]\frac{1}{Z} \tag{4.38}$$

The dynamic transfer function is given by

$$\frac{X_o - X_{o0}}{X_1} = -\frac{\mu}{\dfrac{Z}{R_L} + 2}\,\frac{1}{\dfrac{ZC}{\dfrac{Z}{R_L} + 2}P + 1} \tag{4.39}$$

where Z is obtained from (4.33). The time constant b and the gain k/b of the circuit are, respectively,

$$b = \frac{ZC}{\dfrac{Z}{R_L} + 2} \qquad \frac{k}{b} = -\frac{\mu}{ZC} \tag{4.40}$$

The circuit shown in Fig. 4.15*a* is a variation of the two-pentode integrator shown in Fig. 4.14*a*. If the use of a power supply "floating" with respect to ground is admissible, this circuit permits one to obtain the integrated output voltage at zero reference level with respect to ground.

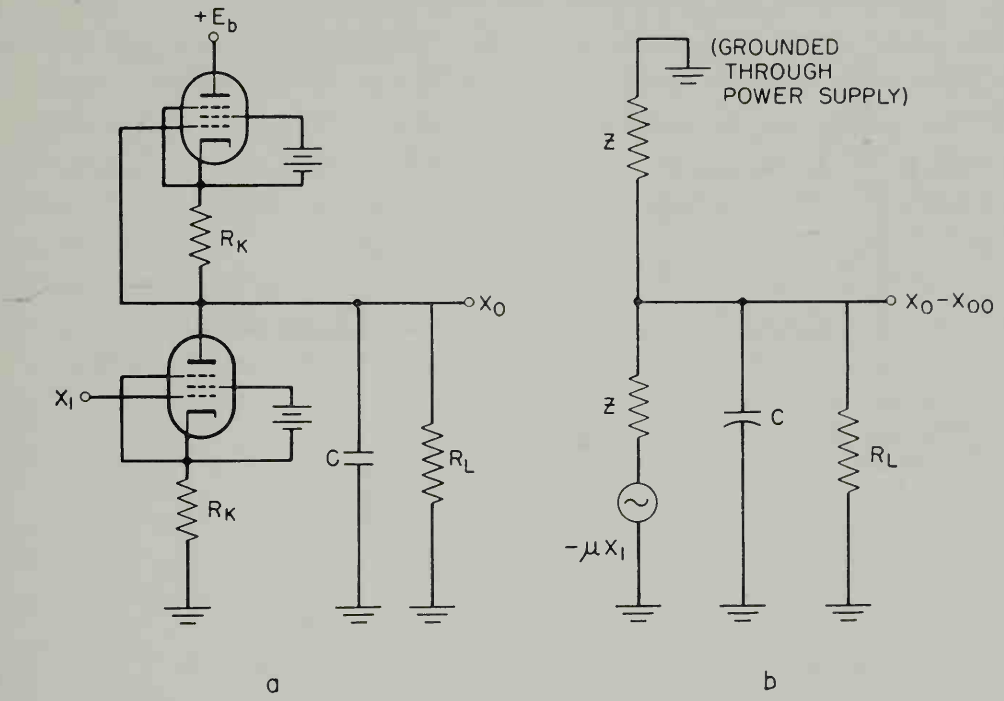

FIG. 4.14. Two-pentode integrator and equivalent circuit.

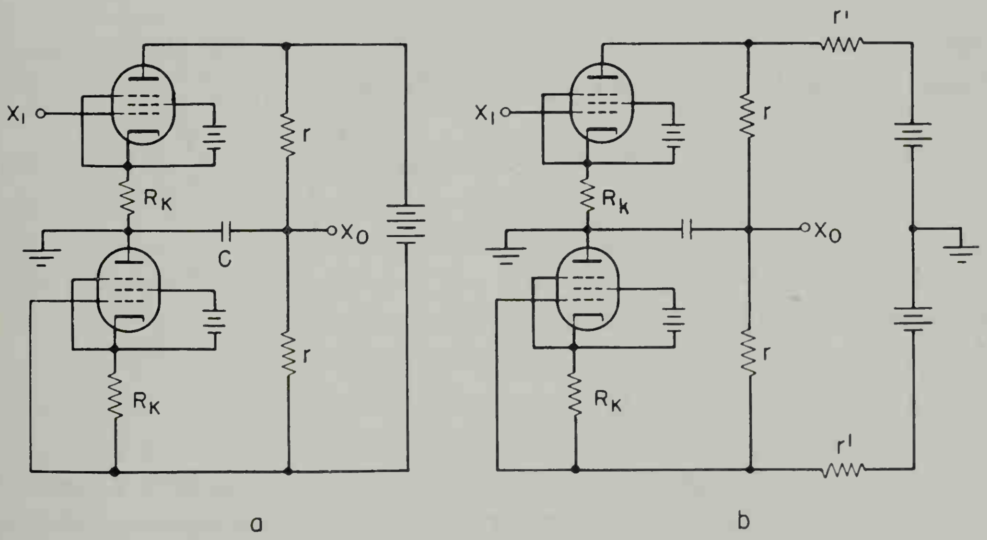

FIG. 4.15. Two two-pentode integrator circuits with output at zero reference level.

The time constant and gain of this circuit are also given by (4.40). The circuit of Fig. 4.15*b* does away with the floating power supply at the expense of integrator accuracy.

In the above circuits, the time constant *b* depends on the value of the cathode resistance, which is necessarily limited to about 100,000 ohms. In the integrator shown in Fig. 4.16, an additional integrating resistor R is added to the cathode resistor R_K of the "resistance amplifier" tube V_1. The second tube V_2 acts again as a voltage amplifier. The *equivalent circuit for voltage changes* is shown in Fig. 4.16*b*, and the time constant b and gain k/b of the circuit are found to be, respectively,

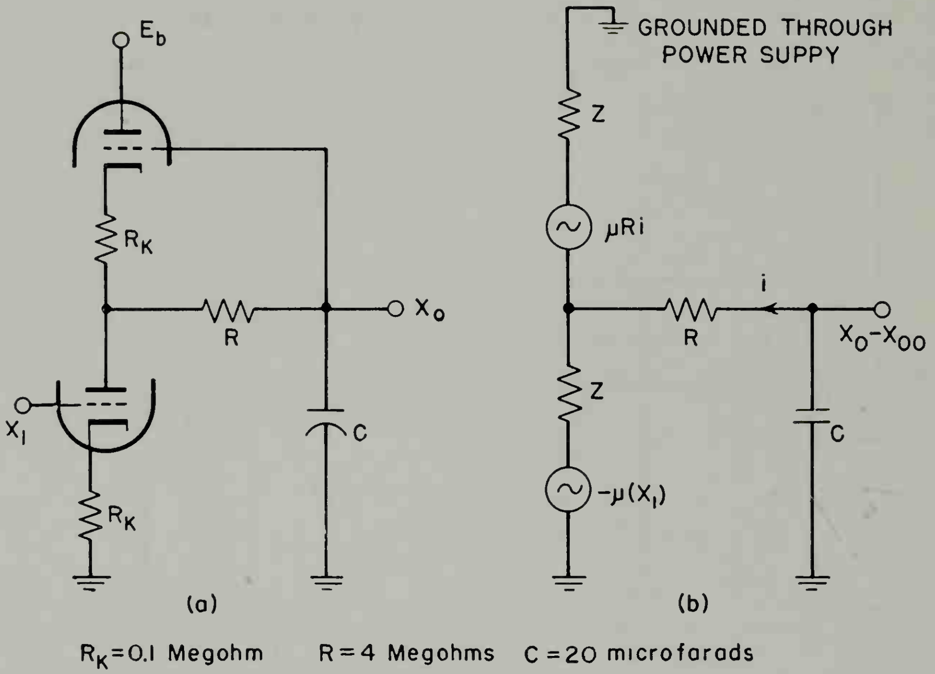

FIG. 4.16. Another integrator circuit based on resistance amplification.

$$\left.\begin{aligned} b &= \tfrac{1}{2}[Z + (2 + \mu)R]C \\ \frac{k}{b} &= \frac{-\mu}{[Z + (2 + \mu)R]C} \end{aligned}\right\} \tag{4.41}$$

The "amplified resistance" is seen to have the value

$$\tfrac{1}{2}[Z + (2 + \mu)R]$$

Whereas the technique of resistance amplification permits substantial increases in the values of the time constant b, all the integrators shown in Figs. 4.12 to 4.16 are severely affected by loading. They should, accordingly, be connected only to loads of high impedance, such as buffer amplifiers, which can also be used to obtain output at zero reference level with respect to ground ($X_{o0} = 0$) where this is necessary. It should be mentioned, finally, that the circuits shown in Figs. 4.14 to 4.16 possess an additional advantage in that the "balanced" arrangement of two similar tubes compensates, in a measure, for certain supply voltage variations. This matter will be discussed in more detail in Chap. 5.

The examples presented do not exhaust the possibilities of resistance amplification; many other circuits and combinations are possible. These circuits may also be combined with some of the other integrating circuits on pages 137–139, although the complications of such a procedure are not ordinarily warranted. In general, the high output impedance of any d-c amplifier with current feedback could be used for integration purposes; theoretically, any desired value of resistance can be obtained in this manner.

Bootstrap Integrator. Another type of integrator, shown in Fig. 4.17, is similar in principle to the bootstrap integrator, originally used as a pulse integrator to provide linear sweep for fire-control radar. In this integrator, a feedback circuit is used to neutralize the effects of the "counter-emf" which arises across the capacitor as the input voltage attempts to charge it through the resistance R. Specifically, this opposing voltage is amplified by a d-c amplifier which must have a positive forward gain A. The amplifier output voltage is added to the integrator input voltage by means of the summing network shown. The input voltage can thus be "boosted" by an amount sufficient to overcome the opposing "emf" of the capacitor. If the amplifier gain is correctly chosen, the resulting voltage across the capacitor C and thus also the amplifier output voltage X_o will be proportional to the time integral of the input voltage X_1.

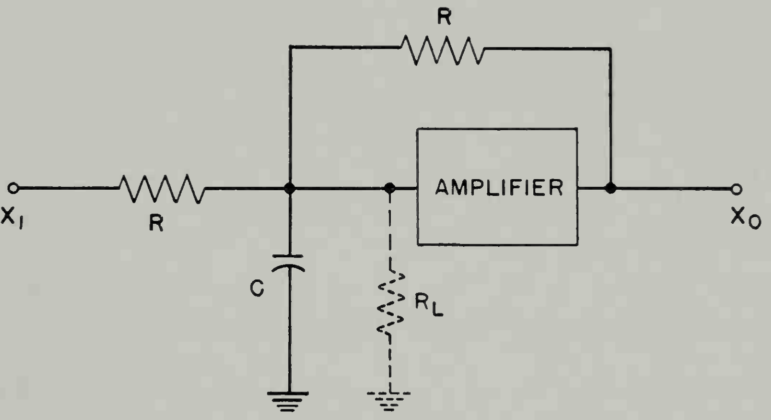

FIG. 4.17. Block diagram of bootstrap integrator.

It should be noted that this integrator does *not* neutralize the voltage opposing the charging voltage by inverse feedback but rather supplies just enough extra voltage "boost" to permit the capacitor to charge in the manner shown in Fig. 4.9.

If the leakage resistance R_L is neglected, the transfer function of the bootstrap integrator in Fig. 4.17 is found (from the nodal equation) to be

$$\frac{X_o}{X_1} = \frac{A}{RCP + (2 - A)} = \frac{A}{(2 - A)\left(\dfrac{RC}{2 - A}P + 1\right)} \tag{4.42}$$

When the leakage resistance is taken into consideration, Eq. (4.42) is replaced by

$$\frac{X_o}{X_1} = \frac{A}{RCP + \left(\dfrac{R}{R_L} + 2 - A\right)} = \frac{A}{\left(\dfrac{R}{R_L} + 2 - A\right)\left(\dfrac{RC}{\dfrac{R}{R_L} + 2 - A}P + 1\right)} \tag{4.43}$$

The time constant of this integrator is seen to be

$$b = \frac{RC}{\dfrac{R}{R_L} + 2 - A} \tag{4.44}$$

and

$$\frac{k}{b} = \frac{A}{RC} \tag{4.45}$$

True integration, or

$$\frac{X_o}{X_1} = \frac{A}{RCP} \tag{4.46}$$

will be obtained if the amplifier gain A and the resistances R and r are adjusted so that

$$\frac{R}{R_L} + 2 - A = 0 \tag{4.47}$$

Little is known about practical applications of this type of integrator at the present time. While the quality of the integration will be sensitive to the gain and resistance adjustments, the use of high-quality resistors and of a stable feedback amplifier should result in good integration. Negative feedback around the amplifier will permit one to obtain a stable, adjustable gain A and will also reduce the voltage appearing on the input grid. Integrators of this type can be used to drive reasonable loads (5,000 to 10,000 ohms) without affecting the quality of the integration.

4.5. The Parallel-feedback Integrator: Error Analysis of a D-c Integrator. *Introduction.* The feedback integrator shown in Fig. 4.18 and briefly described in Sec. 1.4 is the one most generally used in computer applications because of its ability to perform accurate integration with little attenuation and with reasonable values of resistance and capacitance. Again, the quality of integration is not badly impaired by matching the output to a potentiometer of reasonably low impedance (5,000 to 100,000 ohms).

The integrator shown in Fig. 4.18 consists of an integrating feedback network comprising the resistance R and the capacitance C in conjunction with a d-c amplifier of gain A; since the leakage resistance R_L of the capacitor constitutes an additional feedback path, this must also be taken into consideration. An input voltage X_1 will attempt to charge the capacitor plates connected to the amplifier grid. The output voltage X_o of the negative gain d-c amplifier will, however, tend to charge the other capacitor plates so as to neutralize the charge on the input side. The net effect is to keep the grid voltage (which would tend to oppose the input voltage X_1) quite small while the output side of the capacitor is charged to a voltage X_o proportional to the negative time integral of the input voltage X_1.

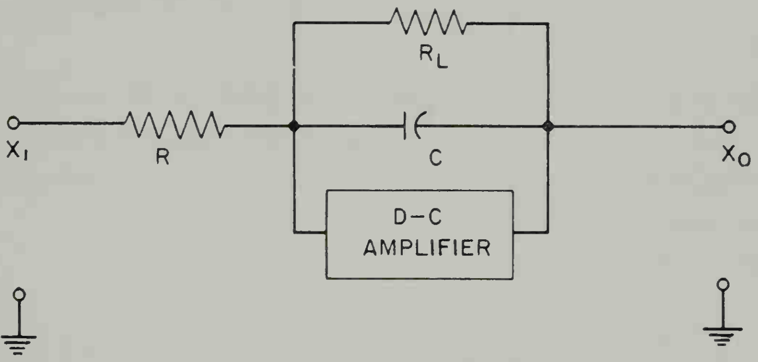

FIG. 4.18. Block diagram of a parallel-feedback integrator.

The purpose of the following analysis is to find integrator designs such that the output voltage X_o of the integrator will be as closely as possible proportional to the time integral X_1/P of the input voltage X_1. Because of the usefulness of the parallel inverse feedback integrator, the latter has been chosen here as the subject of a rather detailed error analysis. Some errors inherent in physical amplifiers (distortion, drift, etc.) are considered as well as the properties of the linear circuits involved.

Performance Equation. In Fig. 4.18, the current flowing through R must be equal to the sum of the currents flowing through R_L and C if grid current is neglected for the time being. The "nodal" equation expressing Kirchhoff's first law is thus

$$\left(\frac{X_o}{A} - X_o\right)\left(CP + \frac{1}{R_L}\right) + \left(\frac{X_o}{A} - X_1\right)\frac{1}{R} = 0 \qquad (4.48)$$

so that

$$X_o = \frac{A}{(1 - A)RCP + (1 - A)\dfrac{R}{R_L} + 1} X_1 \tag{4.49}$$

or

$$\frac{X_o}{X_1} = \frac{A}{(1 - A)\dfrac{R}{R_L} + 1} \frac{1}{\dfrac{(1 - A)RC}{(1 - A)\dfrac{R}{R_L} + 1} P + 1} \tag{4.50}$$

The time constant b and the gain k/b are, in this case,

$$b = \frac{(1 - A)\,RC}{(1 - A)\dfrac{R}{R_L} + 1} \qquad \frac{k}{b} = \frac{A}{(1 - A)RC} \tag{4.51}$$

which makes it possible to obtain substantially better results than with a simple RC network. The expressions (4.51) may be substituted into equations like (4.14), (4.17), (4.20), and (4.23) to give further information on the integrator performance.

It may be seen that true integration, or

$$\frac{X_0}{X_1} = \frac{A}{(1 - A)RCP} \qquad \left(\approx -\frac{1}{RCP} \text{ for } |A| \gg 1\right) \tag{4.52}$$

is approximated most closely for small values of the quantity

$$(1 - A)\frac{R}{R_L} + 1$$

In practice, this can be achieved, for instance, by making the gain A large (from 10^3 to 10^7 at low frequencies) and by using capacitors with as high a leakage resistance R_L as possible. Dielectric absorption acts somewhat like a leakage resistance which is variable with frequency and should also be kept to a minimum. The actual choice of the type of integrating capacitor is discussed in Sec. 5.5.

Integrator Calibration. If the integrator transfer function is taken to be

$$\frac{X_o}{X_1} \approx \frac{A}{(1 - A)RCP} \tag{4.53}$$

the value of $A/[(1 - A)RC]$ may be found by actually measuring the output-to-input voltage ratio by means of a good potentiometer. Equation (4.53) may be replaced by

$$\frac{X_o}{X_1} = -\frac{1}{RCP} \tag{4.54}$$

with a percentage calibration error of magnitude

$$\epsilon_c = \frac{100}{A} \tag{4.55}$$

Careful selection of components makes it possible to determine the desired value of RC in (4.54) within 0.5 per cent.

Effects of Finite Gain and Leakage. A quantitative estimate of the accuracy of (4.53) and thus of the quality of integration is found by substituting the values of b and k from (4.51) into Eq. (4.20). The percentage error ϵ due to finite gain and leakage is found to satisfy the relation

$$\epsilon \leq \left| \frac{50\tau}{(1 - A)RC} \left[(1 - A) \frac{R}{R_L} + 1 \right] \right| \tag{4.56}$$

If finite gain and leakage were the only cause of error, X_o must be within ϵ per cent of the integral form

$$X_o = \frac{A}{(1 - A)RCP} X_1$$

if

$$\tau \leq \frac{1}{50} \frac{\epsilon(1 - A)RC}{(1 - A)\dfrac{R}{R_L} + 1} \tag{4.57}$$

which is a special case of (4.21).

As in Sec. 4.3, this important criterion is *not* restricted to the case of unit step-function input voltages, since the networks in question are linear, and any input voltage $X_1(\tau)$ may be represented as a sum of suitable step functions. *The condition given by* (4.57) *constitutes a limitation on the maximum allowable computing time if the integrator errors are not to exceed a given percentage,* ϵ, *of the output voltage.*

In well-designed *summing amplifiers,* errors due to gain changes and distortion may be made so small that their effects are practically negligible. In electronic integrators, however, even small initial errors may increase owing to the very nature of the integration process. It is, therefore, necessary to estimate all errors as carefully as possible.

Effect of Changes in Gain and Leakage. Changes of the amplifier gain A may affect the error estimates given by Eqs. (4.56) and (4.57). The results of (4.56) and (4.57) may also be affected by changes in the leakage resistance R_L (due to changes in atmospheric moisture, etc.). The relations (4.51), (4.56), and (4.57) should, therefore, always be computed for the most unfavorable values of A and R_L. In any case, errors due to changes in gain and leakage are minimized by making the amplifier gain A as large as possible. Again, it should be noted that the considerations of this paragraph are not restricted to step-function input voltages as long as gain and leakage do not change appreciably during the computation.

The Effect of Distortion. An amplitude distortion of ϵ_D per cent may be interpreted as due to a voltage $\epsilon_D(X_o/100)$ added to the output voltage X_o of an amplifier or as due to a gain change $\epsilon_D A/100$ applied only to certain portions of the signal. The errors due to distortion may be considered as due to gain changes and thus estimated for the case of sinusoidal or step-function input voltages. For more complicated input voltages, the errors are not easy to estimate because of the nonlinear nature of circuits with distortion.

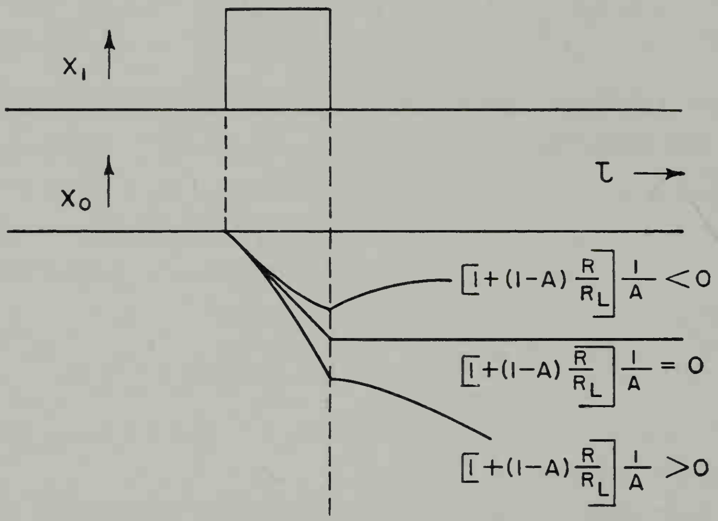
X_1
X_0
τ
$\left[1+(1-A)\frac{R}{R_L}\right]\frac{1}{A}<0$
$\left[1+(1-A)\frac{R}{R_L}\right]\frac{1}{A}=0$
$\left[1+(1-A)\frac{R}{R_L}\right]\frac{1}{A}>0$

FIG. 4.19. Integration of a square voltage pulse.

Qualitatively speaking, the effects of distortion will, in general, be reduced by the feedback; the errors will be minimized by making the forward gain A as large as possible and by reducing τ/RC.

The Effect of Stray Input Voltages. In d-c analog computers, the output of each amplifier is balanced to zero by some biasing arrangement before computation, so that, in the ideal case, only the desired signal voltages can act on the integrator input terminals. Actually, certain stray voltages, due to noise pickup and unbalances in the d-c amplifier resulting from grid current and drift, do occur. They are integrated and may lead to errors in the integrator output voltages.

A noise pulse will be integrated in the manner shown in Fig. 4.19. As long as

$$\left[1 + (1 - A)\frac{R}{R_L}\right]\frac{1}{A} \leq 0$$

high-frequency random noise will tend to cancel itself during the integration process.

D-c integrators are fairly susceptible to errors resulting from pickup of 60-cycle a-c hum. Hum voltages should be kept out of the equipment by reasonable shielding and placement of leads and power supplies which carry alternating currents.

A steady unbalance voltage E_e due to grid current, d-c amplifier drift, etc., at the input of an isolated integrator acts like a step-function input and leads to an absolute output error drift approximately equal to

$$E_e \frac{\tau}{RC} \frac{A}{(1 - A)} \tag{4.58}$$

In order to minimize this drift, which may lead to serious difficulties, it may be expedient to balance each amplifier, not simply for zero output voltage, but rather for zero drift. This will help to eliminate the error given by (4.58). Above all, reduction of integrator drift must be accomplished by careful design of amplifiers and power supplies as discussed in Chap. 5.

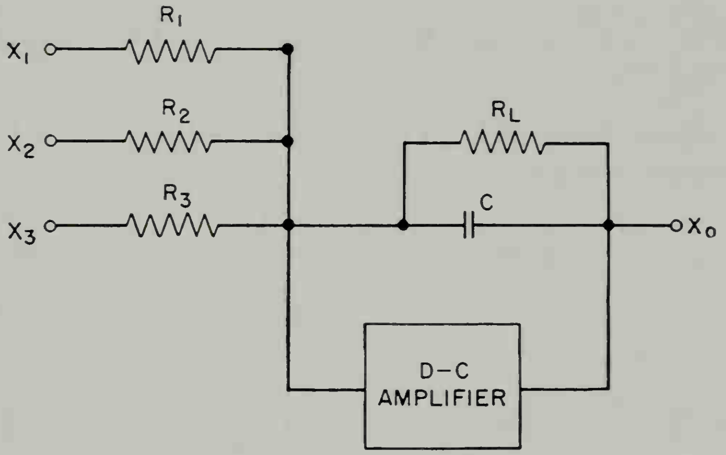

FIG. 4.20. Summing integrator.

In many computer applications, the d-c integrators are not isolated but are set up in feedback loops to simulate stable, damped oscillations. In such circuits, any temporary unbalance will tend to damp out, and any steady unbalance voltage will only change the zero point or reference level of the output voltage scale. The amplifiers in such a system need only be balanced for zero output at zero input voltage.

Summing Integrators. It was shown in Sec. 1.4 that summing and multiplication by constants may be achieved at the input of an integrator. Such a summing integrator is shown in Fig. 4.20. The output voltage of this circuit is given by

$$X_o = \sum_{i=1}^{n} \frac{X_i}{R_i} \frac{A}{(1 - A)CP + \dfrac{1 - A}{R_L} + \sum\limits_{i=1}^{n} \dfrac{1}{R_i}} \tag{4.59}$$

or approximately by

$$X_o = -\sum_{i=1}^{n} \frac{X_i}{R_i CP} \tag{4.60}$$

if the voltages $X_1, X_2, \ldots, X_n$ are applied to the input terminals. *Equation* (4.60) *is commonly used for the calibration of summing integrators.* It may be shown that *the error analysis expressed in Eqs.* (4.51), (4.56), *and* (4.57) *will apply to the summing integrator if, in each equation, R is replaced by* $\left(\sum\limits_{i=1}^{n} \dfrac{1}{R_i}\right)^{-1}$, which is the equivalent resistance of the summing resistors R_i taken in parallel. The equivalent *time constant* for the summing integrator is thus

$$b = \frac{(1 - A)CR_L}{(1 - A) + R_L \sum\limits_{i=1}^{n} \dfrac{1}{R_i}} \tag{4.61}$$

In particular, Eq. (4.57) for the maximum computing time allowable for a given percentage error ϵ becomes

$$\tau \leq \frac{1}{50} \frac{\epsilon(1 - A)R_L C}{(1 - A) + R_L \sum\limits_{i=1}^{n} \dfrac{1}{R_i}} \tag{4.62}$$

Effect of D-c Amplifier Input Resistance. The effect of the finite input resistance R_g of the integrating amplifier has hitherto been neglected. This resistance (see Fig. 4.21) may be considered simply as a grounded input terminal in a summing integrator. The results of the last paragraph show that, as far as the quality of integration is concerned, R_g decreases the effective value of R as if it were connected in parallel with R. Accordingly, *it is desirable to keep the value of R_g as high as possible,* and if possible, the use of cathode-follower input to the amplifier is indicated.

Effects of Finite Source and Load Impedances. As with the summing amplifier, the effects of source and load impedances on the integrator must be taken into account in order to predict its performance in actual computations. This subject will be treated in Sec. 4.7.

High-frequency Performance Requirements. The high-frequency performance requirements for the d-c amplifiers used in parallel-feedback integrators are similar to those for summing amplifiers (Sec. 4.2), but they are somewhat less stringent owing to the low high-frequency response needed in integrators.

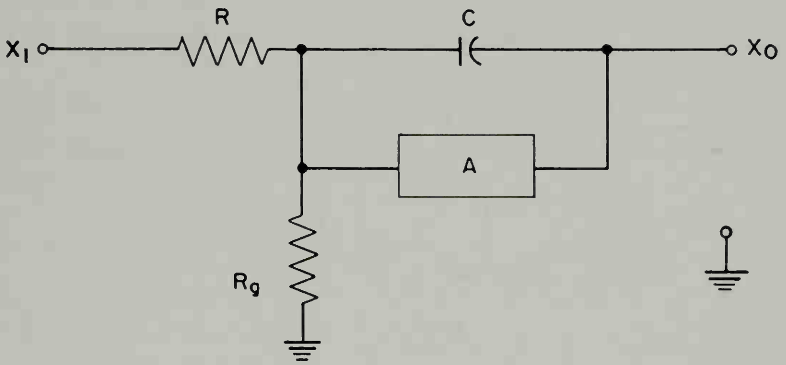

FIG. 4.21. Effect of d-c amplifier input resistance.

1. The absolute value of the d-c amplifier forward gain A must satisfy the condition (4.56) throughout the anticipated range of signal frequencies for each given permissible value of the percentage error ϵ.
2. The integrator must be stable; this will be the case if the roots of the equation

$$[1 - A(P)]RCP + [1 - A(P)]\frac{R}{R_L} + 1 = 0$$

   have only negative real parts.
3. The integrator phase shift should not deviate from 90 deg by more than 0.1 deg throughout the frequency range used.

As in the case of summing amplifiers, the gains and phase shifts of several cascaded integrators may add in certain computer setups and lead to instability.

Conclusions. Most errors in the electronic integrators discussed in this section are reduced by using a shorter computing time, high gain, high leakage resistance, and a long integrator time constant.

The errors due to finite gain, gain changes, and distortion can be made very small by using high values of the d-c amplifier gain A and certain types of feedback amplifiers. The most serious errors, then, arise from *drift* due to unbalances in d-c amplifiers. These errors must be minimized by carefully balancing the amplifiers, by using well-regulated power supplies, and by improved amplifier design. Errors may also be reduced by decreasing τ/RC within the limits set by scale-factor considerations.

4.6. General Discussion of Operational Amplifiers and Networks.

Introduction. In many applications of d-c analog computers, it is feasible to effect considerable reductions in the number of integrators and amplifiers needed to set up a given problem. This is made possible by utilizing special setups which are not directly related to the differential analyzer techniques described in Chaps. 1 and 2 but well applicable to d-c analog computers.

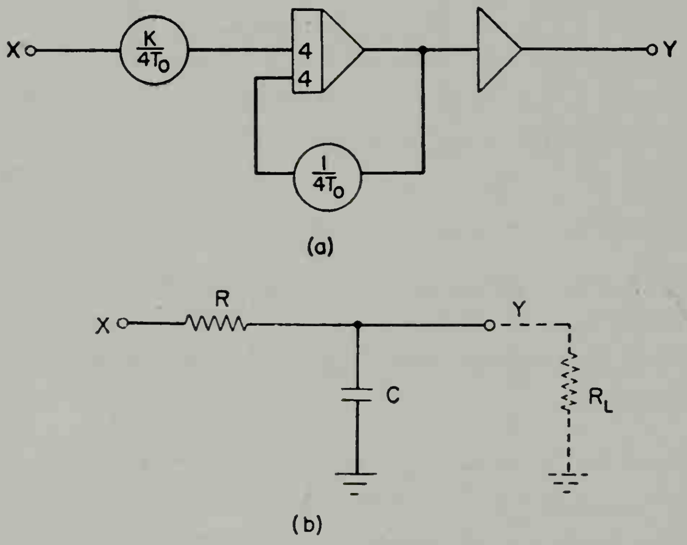

FIG. 4.22. Two possible computer setups for a relation of the form

$$X = \frac{1}{k}(1 + T_0P)Y \qquad (T_0 > 0)$$

Special Computing Networks. As a first example, consider a machine variable (voltage) Y which is related to another machine variable X by the differential equation

$$X = \frac{1}{k}(1 + T_0P)Y \qquad (T_0 > 0) \tag{4.63}$$

Such relations are of importance in many problems describing damping or decay phenomena. Figure 4.22*a* shows how such a relation may be established through the use of conventional differential analyzer techniques; two amplifiers are used. In many cases, it may be useful to save the amplifiers (and thus to avoid any drift and output limitations associated with them) by using the simple network shown in Fig. 4.22*b*. The resistance R_L loading the network (usually the input resistor of an operational amplifier) must be considered as part of the network. According to Eq. (4.12), the network will make the voltage Y vary with X in the manner prescribed by the desired relation (4.63), with

$$T_0 = \frac{RR_LC}{R_L + R} \quad (\approx RC \text{ for } R_L \gg R) \qquad k = \frac{R_L}{R + R_L} \quad (\approx 1 \text{ for } R_L \gg R) \tag{4.64}$$

The value of the "time constant" T_0 is determined by the values of R, R_L, and C. It might be interesting to note that the transfer function

$$\frac{Y}{X} = \frac{k}{(T_0P + 1)} \tag{4.65}$$

just obtained is often useful as a first approximation to the transfer function of simple servomechanisms and data transmission systems possessing a "time delay" T_0.

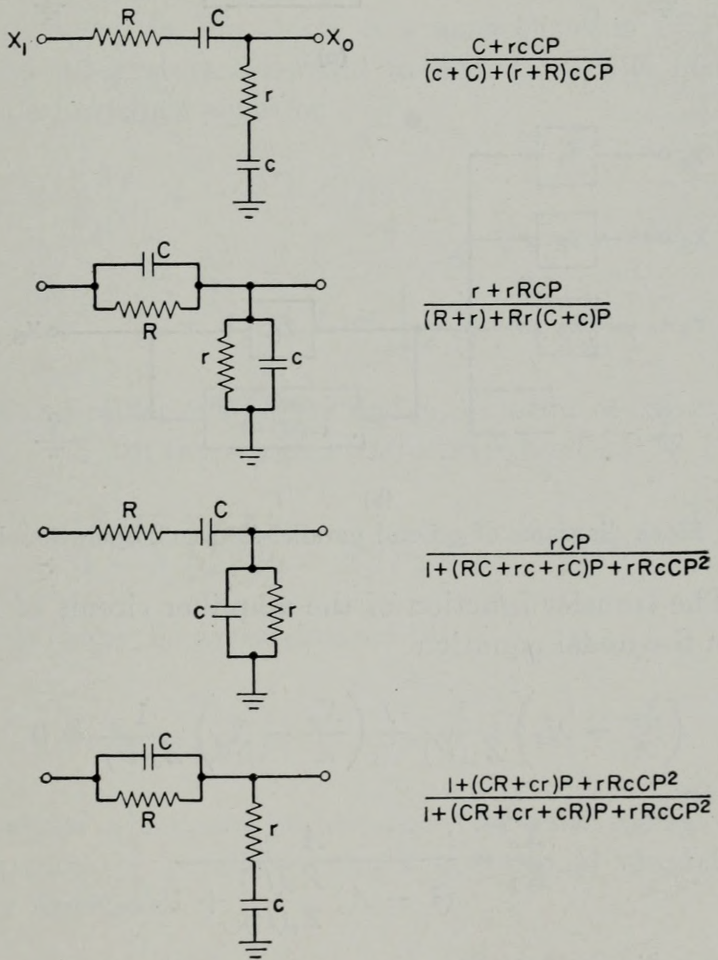
R
C
X_I
X_O
r
c
$\frac{C+rcCP}{(c+C)+(r+R)cCP}$
C
R
r
c
$\frac{r+rRCP}{(R+r)+Rr(C+c)P}$
R
C
c
r
$\frac{rCP}{1+(RC+rc+rC)P+rRcCP^2}$
C
R
r
c
$\frac{1+(CR+cr)P+rRcCP^2}{1+(CR+cr+cR)P+rRcCP^2}$

FIG. 4.23. Some computing networks and their transfer functions. The transfer functions of several simpler networks may be obtained from those given by considering one or more resistances and/or capacitances as equal to zero or infinity.

Figure 4.23 shows a number of other computing networks and their transfer functions.

Parallel-feedback-type Operational Amplifiers. An entire new field of interesting possibilities is opened by a generalization of the operational amplifier technique already applied to electronic integration, differentiation, and addition. The parallel-feedback-type differentiators, integrators, and summing amplifiers discussed in the previous sections may be considered as special cases of the general operational amplifier circuits shown in block-diagram form in Fig. 4.24. The forward gain A of the d-c amplifiers shown is real and negative over the range of working frequencies. The transfer function of the amplifier circuit of Fig. 4.24*a* is derived from the nodal equation

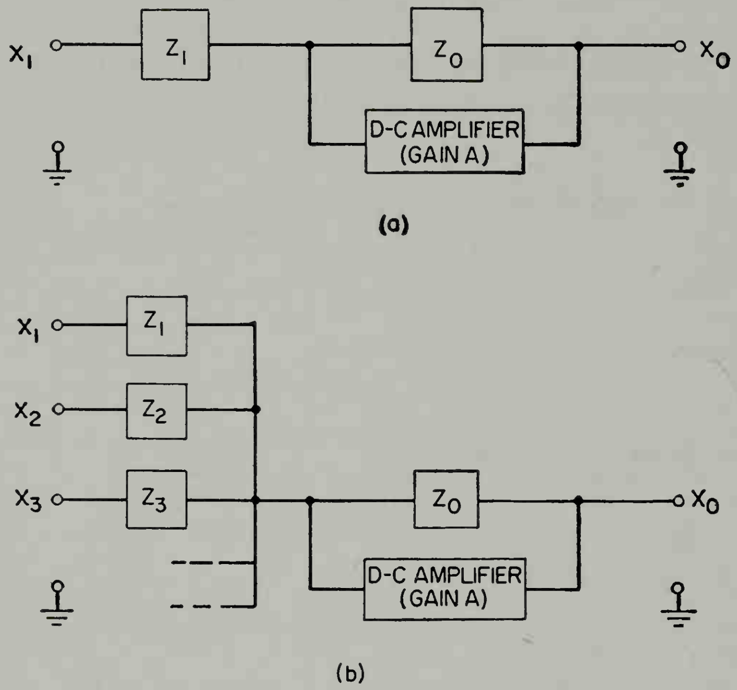

FIG. 4.24. Block diagrams of general parallel-feedback operational amplifiers.

$$\left(\frac{X_o}{A} - X_1\right)\frac{1}{Z_1(P)} + \left(\frac{X_o}{A} - X_o\right)\frac{1}{Z_o(P)} = 0 \tag{4.66}$$

which leads to

$$\frac{X_o}{X_1} = \frac{A}{(1 - A)\dfrac{Z_1(P)}{Z_o(P)} + 1} \tag{4.67}$$

This latter equation reduces to

$$\frac{X_o}{X_1} \approx -\frac{Z_o(P)}{Z_1(P)} \tag{4.68}$$

if $\left|(1 - A)\dfrac{Z_1(P)}{Z_o(P)}\right|$ is large compared with unity for the working frequencies in question.[1] The transfer function expressed by Eqs. (4.67) and (4.68) is clearly a generalization of the previously derived transfer functions of parallel-feedback integrators and phase inverters. The simplified transfer function (4.68) may hold true for some input frequencies but not for others; *an accuracy analysis similar to that presented in Sec.* 4.5 *for the parallel-feedback integrator will be necessary in each case.*

Many interesting applications of such circuits are possible.[2] In particular, if Eq. (4.68) holds, this expression is very convenient to use for approximating many transfer functions useful especially in the solution of problems in automatic control engineering, since a close analogy exists between linear servomechanisms and feedback amplifiers like the one shown in Fig. 4.24. A number of practically useful examples are presented in tabular form in Fig. 4.25.

The circuit shown in Fig. 4.24*b* is a generalization of the summing amplifiers and integrators described in Sec. 1.3. The nodal equation leads to the performance equation

$$X_o = \left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots\right] Z_o(P) \frac{A}{(1 - A) + Z_o(P)\left[\dfrac{1}{Z_1(P)} + \dfrac{1}{Z_2(P)} + \cdots\right]} \tag{4.69}$$

describing the amplifier output voltage X_o in terms of the input voltages $X_1, X_2, \ldots$. If, for the working frequencies used, $|A| \gg 1$ and

$$\left|\frac{Z_o(P)}{1 - A}\left[\frac{1}{Z_1(P)} + \frac{1}{Z_2(P)} + \cdots\right]\right| \ll 1 \tag{4.70}$$

then Eq. (4.69) may be approximated by

$$X_o = -\left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots\right] Z_o(P) \tag{4.71}$$

An error analysis will again be necessary for each special case. Some general properties of parallel-feedback operational amplifiers are discussed in the Appendix.

---

[1] For the purposes of steady-state network analysis, each impedance operator $Z(P)$ may be considered as a complex impedance function $Z(j\omega)$ of the circular frequency ω. This follows if sinusoidal voltages of the form $A \cos (\omega\tau + \alpha)$ are substituted in the circuit equations of the network in question.

[2] Ragazzini, Randall, and Russell, Analysis of Problems in Dynamics by Electric Circuits, *Proc. IRE,* **35:** 444, 1947.

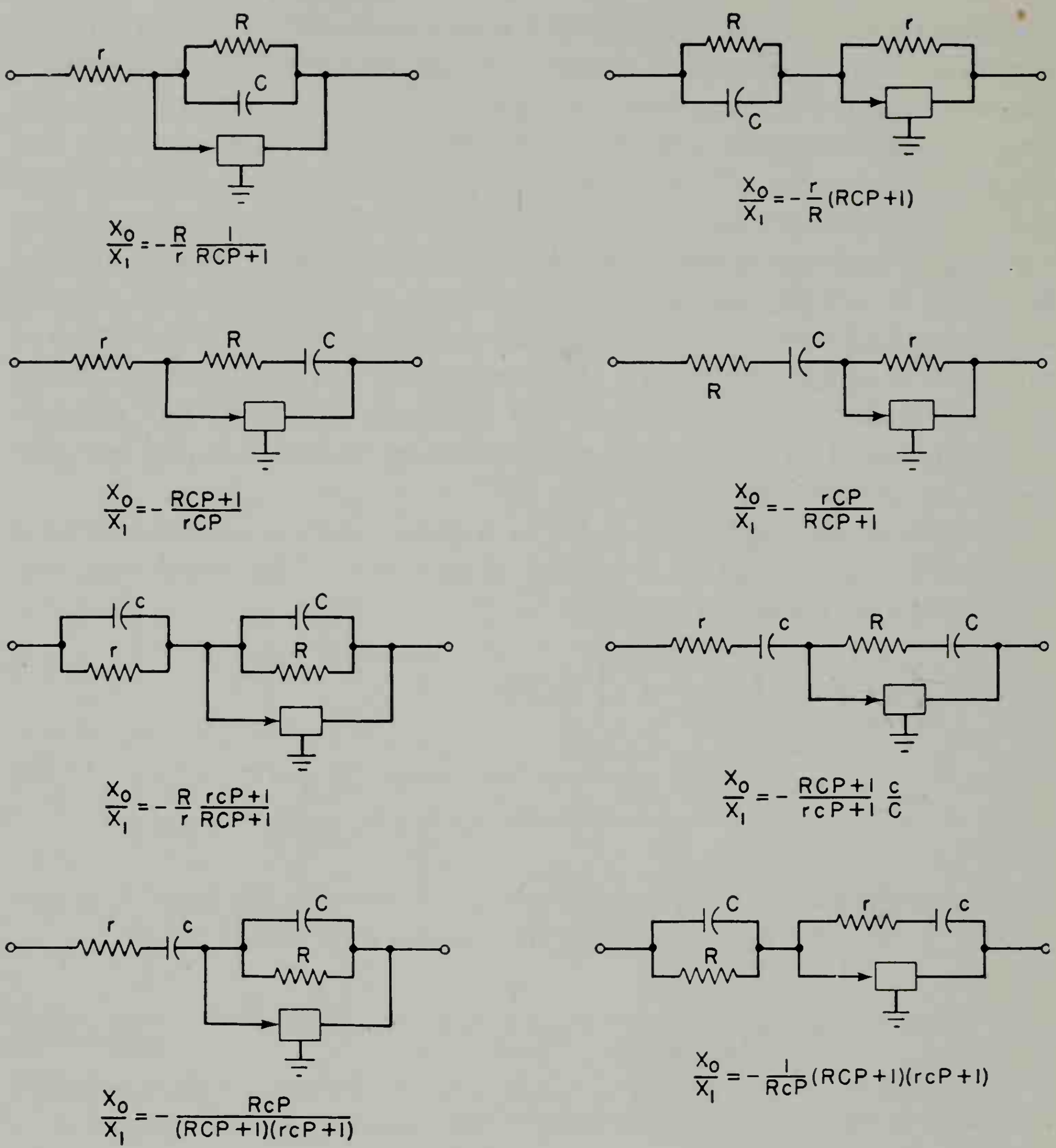
r
R
C
Xo/X1 = - R/r · 1/(RCP+1)
R
C
r
Xo/X1 = - r/R (RCP+1)
r
R
C
Xo/X1 = - (RCP+1)/(rCP)
R
C
r
Xo/X1 = - rCP/(RCP+1)
c
r
C
R
Xo/X1 = - R/r · (rcP+1)/(RCP+1)
r
c
R
C
Xo/X1 = - (RCP+1)/(rcP+1) · c/C
r
c
C
R
Xo/X1 = - RcP/((RCP+1)(rcP+1))
C
R
r
c
Xo/X1 = - 1/(RcP) (RCP+1)(rcP+1)

FIG. 4.25. Table of simplified transfer functions

$$\frac{X_o}{X_1} = -\frac{Z_o(P)}{Z_1(P)}$$

for various operational amplifier circuits.

The Use of High-gain Amplifiers for Implicit Computation. The use of high-gain d-c amplifiers for implicit computation may be considered as a further generalization of the operational amplifier technique and is *not* restricted to linear operations.

Figure 4.26 shows a computer setup which is sometimes useful for solving a machine equation of the type

$$F(X_1X_2, \ldots, X_n,X) = 0 \tag{4.72}$$

for the machine variable X, where $X_1, X_2, \ldots, X_n$ are other machine variables. Inspection of Fig. 4.26 shows that the relation actually enforced is

$$F(X_1,X_2, \ldots, X_n,X) = -\frac{X}{A}$$

but for large values of the d-c amplifier gain A the resulting error will be negligible. The feedback loop shown in Fig. 4.26 will be degenerative and

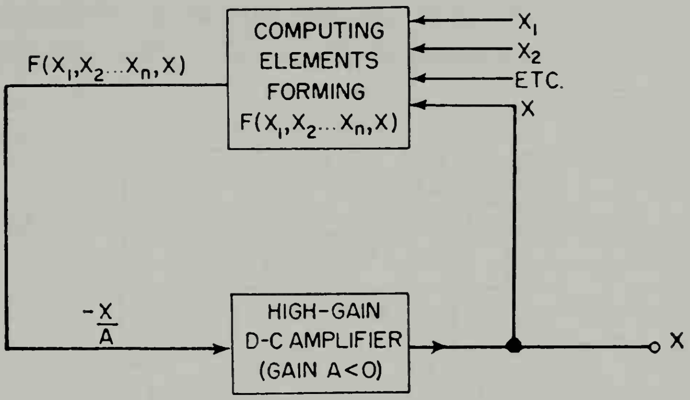
COMPUTING ELEMENTS FORMING F(X1,X2...Xn,X)
F(X1,X2...Xn,X)
X1
X2
ETC.
X
-X/A
HIGH-GAIN D-C AMPLIFIER (GAIN A<0)
X

FIG. 4.26. Use of a high-gain amplifier for implicit solution of the machine equation

$$F(X_1, X_2 \ldots X_n, X) = 0 \text{ with } \frac{\partial F}{\partial X} > 0$$

tend to be stable if there is only one solution $X = X(X_1,X_2, \ldots, X_n)$ of Eq. (4.72) and if[1]

$$A \frac{\partial F}{\partial X} < 0 \tag{4.73}$$

If the sign of the derivative $\partial F/\partial X$ changes as the machine variables $X_1, X_2, \ldots X_n$ change, X can frequently be computed by solving the machine equation

$$F \frac{\partial F}{\partial X} = 0 \tag{4.74}$$

instead of Eq. (4.72); Eq. (4.74) constitutes a necessary condition that $[F(X_1,X_2, \ldots, X_n,X)]^2$ be a minimum. The method of implicit computation by means of high-gain amplifiers may be still further generalized to permit the implicit solution of ordinary differential equations and of sets of simultaneous equations. The important case of simultaneous linear equations is discussed in Sec. 3.9. A number of other applications are discussed in Secs. 6.3 and 6.5.

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[1] Greenwood, I. A., J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948.

The high-frequency stability of computer setups like that of Fig. 4.26 *will require careful investigation in each case.* In general, the loop gain of the feedback loop will vary with the machine variables $X_1, X_2, \ldots, X_n$. The design of the loop must, therefore, necessarily be a compromise; for this reason, the special implicit methods just described should never be used for accurate computation without the most careful error analysis.

Other Types of Operational Amplifiers. The operational amplifiers hitherto discussed in this section were all of the *parallel-feedback* type. It is, however, possible to construct operational amplifiers based on *series feedback* in the manner shown in Fig. 4.27. The integrator circuit of Fig. 4.12 is an example of an operational amplifier using series feedback. The circuit of Fig. 4.27[1] has the transfer function[2]

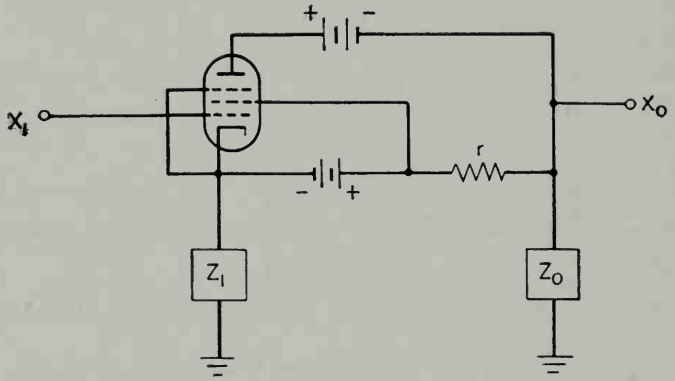

FIG. 4.27 Block diagram of an operational amplifier based on series feedback (G.A. Philbrick, *op. cit.*), This circuit has been improved by McCool (*op. cit.*), who replaced the screen resistor r by the dynamic plate resistance of a second pentode (see Sec. 4.4).

$$\frac{X_o}{X_1} = -\frac{Z_o(P)}{Z_1(P)}\,\frac{g_m}{g_m + \left[1 + \dfrac{Z_o(P)}{Z_1(P)}\right]\left(\dfrac{1}{r} + \dfrac{1}{r_p}\right) + \dfrac{1}{Z_1(P)}}$$

$$\left\{\approx -\frac{Z_o(P)}{Z_1(P)} \text{ for } \left|\frac{1}{g_m}\left[\left(1 + \frac{Z_o}{Z_1}\right)\left(\frac{1}{r} + \frac{1}{r_p}\right) + \frac{1}{Z_1}\right]\right| \ll 1\right\} \qquad (4.75)$$

where g_m and r_p are the transconductance and plate resistance, respectively, of the tube used. The relatively high output impedances associated with series feedback tend to make such circuits less useful for conventional applications than the corresponding parallel-feedback-type operational amplifiers.

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[1] Philbrick, *op. cit.*

[2] McCool, *op. cit.*

Finally, it might be interesting to investigate the simultaneous application of parallel and series feedback to operational amplifiers and also the design of operational amplifiers for computers in which the machine variables representing mathematical quantities are currents rather than voltages.

4.7. Regenerative Operational Amplifiers. *Reduction of the Effects of Distortion and Instability through the Use of Feedback.* One might think that the application of degenerative feedback might decrease the effects of distortion and gain changes and, on the other hand, regeneration could increase the effective forward gain A of an operational amplifier like that of Fig. 4.24b and thus make its performance equation correspond more closely to the desired form

$$X_o = -\left(\frac{X_1}{Z_1} + \frac{X_2}{Z_2} + \cdots\right)Z_o \tag{4.76}$$

A closer investigation shows that at least the first of these possibilities has hardly any practical value, as the decrease in forward gain due to degenerative feedback would tend to offset the possible advantages of the degeneration. As a matter of fact, it is shown in the Appendix that, whenever

$$\left|\frac{Z_0}{(1 - A)}\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)\right| \ll 1 \tag{4.77}$$

the percentage errors due to distortion and gain changes in the operational amplifier of Fig. 4.24b will be determined by the quantities ϵ_D/A and ϵ_A/A, respectively, where ϵ_D and ϵ_A are the respective percentages of distortion and gain changes in the d-c amplifier used. This implies that *percentage errors due to distortion and gain changes in linear operational amplifiers will not be decreased or increased through the application of* OVER-ALL *feedback* (*either degenerative or regenerative*) *around the d-c amplifier used as long as* (4.77) *holds.*

As a result of these considerations, it may be said that a certain amount of over-all regeneration can be used to increase the forward gain of an operational amplifier without directly increasing the effects of distortion and gain changes.

Application of Feedback around Sections of the D-c Amplifier. More interesting results can be achieved through the use of feedback around stages or groups of stages of multistage d-c amplifiers.[1] The amplifier shown in Fig. 4.28 will serve as an illustrative example. It comprises a voltage amplifier section of gain a_1 with a feedback loop of feedback ratio

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[1] Korn, G. A., D-c Integrator Design, *Electronics*, May, 1948.

β_1 and a power amplifier section of gain a_2 with a feedback loop of feedback ratio β_2. The over-all gain A is

$$A = \frac{a_1 a_2}{(1 - a_1\beta_1)(1 - a_2\beta_2)} \tag{4.78}$$

This will be the forward gain of an operational amplifier utilizing the d-c amplifier in question. If ϵ_{a1} and ϵ_{a2} are percentage gain changes in the two sections of the d-c amplifier, respectively, before the application of feedback, the corresponding percentage gain change ϵ_A of the entire d-c amplifier is

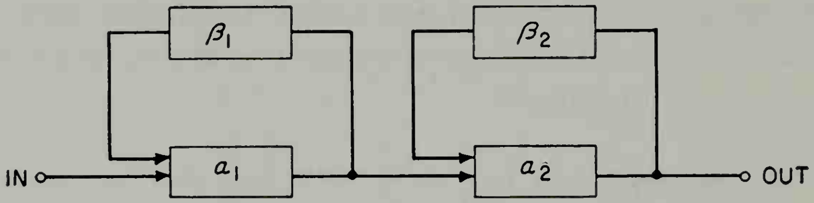

FIG. 4.28. Block diagram of a d-c amplifier with feedback around sections of the amplifier.

$$\epsilon_A = \frac{\epsilon_{a1}}{1 - a_1\beta_1} + \frac{\epsilon_{a2}}{1 - a_2\beta_2} \tag{4.79}$$

so that

$$\frac{\epsilon_A}{A} = \frac{1}{a_1 a_2}[\epsilon_{a1}(1 - a_2\beta_2) + \epsilon_{a2}(1 - a_1\beta_1)] \tag{4.80}$$

Similarly, if ϵ_{d1} and ϵ_{d2} are the distortion percentages of the two amplifier sections, without feedback, the corresponding distortion percentage of the entire d-c amplifier is

$$\epsilon_D = \frac{\epsilon_{d1}}{1 - a_1\beta_1} + \frac{\epsilon_{d2}}{1 - a_1\beta_2} \tag{4.81}$$

and thus

$$\frac{\epsilon_D}{A} = \frac{1}{a_1 a_2}[\epsilon_{d1}(1 - a_2\beta_2) + \epsilon_{d2}(1 - a_1\beta_1)] \tag{4.82}$$

The expressions (4.80) and (4.82) may be introduced into the analysis of the effects of gain changes and distortion in high-gain operational amplifiers derived in the Appendix. This analysis leads to the conclusion that with many operational amplifier circuits *the effects of gain changes and distortion in the power amplifier section can be made very small through the use of near-critical regeneration in the voltage amplifier section, as well as by high stage gains in all stages.* More generally, the effects of gain changes and distortion in any stage can be reduced through the application of regenerative feedback to other stages or groups of stages of the d-c amplifier.

The results just derived indicate the use of regenerative feedback around sections of multistage d-c amplifiers as a very useful method for increasing the accuracy of operational amplifiers. Such feedback will tend to make the performance equation correspond more closely to the desired form (4.76), will reduce the output impedance of parallel-feedback operational amplifiers, and will help to minimize the effects of distortion and gain changes in the d-c amplifier.

Before deciding on the application of regenerative feedback, one must ascertain (1) whether the circuits will be stable (see Sec. 5.1) and (2) by how much the "return differences" $(1 - a_1\beta_1)$, $(1 - a_2\beta_2)$, etc., can change from the desired small values as the result of changes in the stage gains and feedback network parameters. It is highly desirable to design the amplifier so that frequent readjustments of stage gains or feedback ratios are not necessary.

A good example of a d-c amplifier incorporating internal regenerative feedback is shown in Fig. 5.32. Other examples have been discussed by Valley and Wallman.[1]

Feedback around one or more stages of the d-c amplifier is also used at times for purposes of stabilizing the high-frequency response of an operational amplifier (see Sec. 5.4). An example of such an application is shown in Fig. 5.39.

Regenerative Integrators. If the absolute value of the negative forward gain A of a parallel-feedback integrator of the type shown in Fig. 4.18[2] is made very large through regeneration or other means, the transfer function

$$\frac{X_o}{X_1} = \frac{A}{(1 - A)RCP + (1 - A)\dfrac{R}{R_L} + 1} \tag{4.83}$$

derived in Sec. 4.5 becomes

$$\frac{X_o}{X_1} = -\frac{R_L}{R}\frac{1}{R_LCP + 1} \tag{4.84}$$

The "leakage time constant" R_LC seems to constitute an upper limit for

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[1] Valley, G. E., and H. Wallman, *Vacuum Tube Amplifiers*, MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948.

[2] The analysis of this section applies to *summing integrators* of the type shown in Fig. 4.20 if the quantity R in (4.83) and the following equations is replaced by

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots\right)^{-1}$$

the time constant (see Sec. 4.5) of the parallel-feedback integrator. This limitation is due to degenerative feedback through the leakage resistance.

An infinitely large integrator time constant or perfect integration could be realized if it were possible to make the quantity

$$(1 - A)\frac{R}{R_L} + 1 \tag{4.85}$$

equal to zero throughout the range of working frequencies of the integrator. It is seen that the expression (4.85) can be equal to zero only for positive values of the forward gain A.

Unfortunately, this condition for perfect integration cannot be satisfied by simply using any d-c amplifier whose forward gain A is positive instead of the usual negative-gain amplifiers. Since the phase shift of the integrator feedback network is practically equal to 90-deg lead at high frequencies, *any high-frequency phase lag in an ordinary d-c amplifier of positive gain would result in regeneration and tend to produce instability at high frequencies;* the resulting circuit would act essentially like a multivibrator[1] unless special precautions were taken to equalize the high-frequency response.

It can be shown[2] that *the use of regeneration either around the entire d-c amplifier or (preferably) around a section of the d-c amplifier makes it possible to obtain a positive d-c amplifier gain and thus to make the quantity* (4.85) *equal to zero for perfect integration without making the integrator unstable at high frequencies.* The technique just described can be a striking means for improving the performance of d-c integrators by overcoming the effects of finite amplifier gain and capacitor leakage.

The adjustment of a regenerative integrator will be sensitive not only to changes of the forward gain A of the amplifier but to variations in the leakage resistance R_L as well. Changes in leakage resistance due to changes in atmospheric moisture may be hard to avoid if the values of R_L are very large. It is clear that, due to unavoidable changes in gain and leakage, it will not be possible to achieve perfect integration for an indefinite time. Nevertheless, considerable improvements in integrator performance can be obtained. The error estimates for parallel-feedback integrators derived in Sec. 4.5 hold for regenerative integrators provided that the quantity (4.85) is less than or equal to zero. In order to obtain realistic error estimates for regenerative integrators, it is necessary to know only between what limits the gain A and the leakage resistance R_L

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[1] Greenwood, Holdam, and MacRae, *op. cit.*

[2] See Bode, *op. cit.* The method of analysis is essentially similar to that employed in Sec. 5.4.

may vary during computation. The error estimates derived in Sec. 4.5 can then be applied to the most unfavorable combination of A and R_L.

Figure 4.19 shows how a pulse is integrated by a parallel-feedback integrator when the quantity (4.85) is positive, negative, and equal to zero.

It is clear that a perfect d-c integrator can be realized *only* through the use of regeneration, since the perfect integrator must necessarily have infinite gain at zero frequency.

It is possible to apply similar feedback techniques to electronic integrators other than the parallel-feedback integrators discussed here; the bootstrap integrator described in Sec. 4.4 is an example of such a circuit.

CHAPTER 5

D-C AMPLIFIERS FOR COMPUTER APPLICATIONS

5.1. Principles of D-c Amplifier Operation. *Introduction.* In *direct-coupled or d-c amplifiers,* finite gain at zero frequency is obtained through conductive coupling between stages. Such conductive coupling may involve direct connections or resistive interstage networks; no coupling transformers or "blocking" capacitors are used.[1] The output voltage of a d-c amplifier or of any individual stage will, thus, be sensitive to changes in the input d-c reference level or input voltage for zero input signal. Connections to the amplifier input and output terminals are usually made also through conductive coupling except in a few instances such as in certain differentiating circuits.

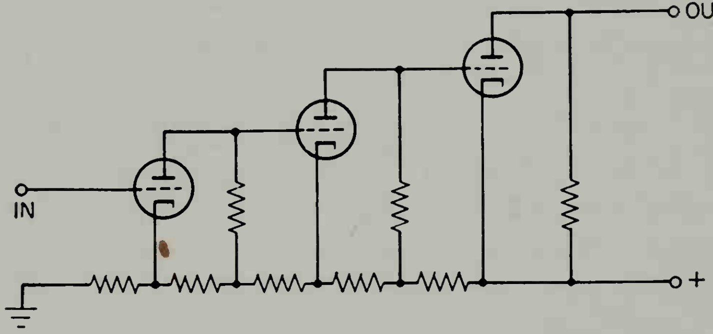

FIG. 5.1. D-c amplifier stages cascaded along a common bleeder.

Cascading of D-c Amplifier Stages. The special nature of d-c amplifiers gives rise to a number of special design problems. The first among these is based on the fact that the voltage at the plate of each amplifier tube is necessarily higher than that at the grid. This makes the *cascading of amplifier stages* more difficult than in a-c amplifiers. It is possible to raise the cathode of the succeeding amplifier stage to a potential sufficiently high to obtain a correct grid bias from the plate voltage of the first stage. Such a procedure requires high power-supply voltages if several stages are to be cascaded; Fig. 5.1 shows how the correct supply voltages for several stages may be obtained from a common bleeder resistor.

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[1] Valley, G. E., and H. Wallman, *Vacuum Tube Amplifiers,* Chap. 11, Direct-coupled Amplifiers by J. Gray, MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948.

A number of alternative designs, all essentially based on the circuit shown in Fig. 5.2, permit *operation of several d-c amplifier stages from the same power-supply terminals.* By means of a resistive summing network comprising the resistances R_1 and R_2 (either of which may be replaced by vacuum or gas tubes in certain designs) a negative bias voltage E_c is added to the voltage appearing at the plate of the amplifier tube. In this manner any desired d-c reference level may be obtained at the output of the stage at the price of some signal attenuation due to the summing network.

Balancing of Computer Amplifiers. The output d-c reference level of practically all d-c amplifiers used in d-c analog computing elements must be equal to the input d-c reference level (usually zero with respect to ground) in order to make it possible to apply feedback from amplifier output terminals and to interconnect the various computing elements. The desired output reference level must be obtained by means of circuit arrangements similar to that shown in Fig. 5.2 (see page 181). Accordingly, the d-c amplifiers used in d-c analog computers will in general require at least two supplies. In each case, it will be necessary to adjust one or more of the resistors determining the voltage levels (see Figs. 5.29 to 5.35) carefully so as to obtain zero output voltage when the input voltage is zero. This process is called *balancing* or *zero adjustment.*

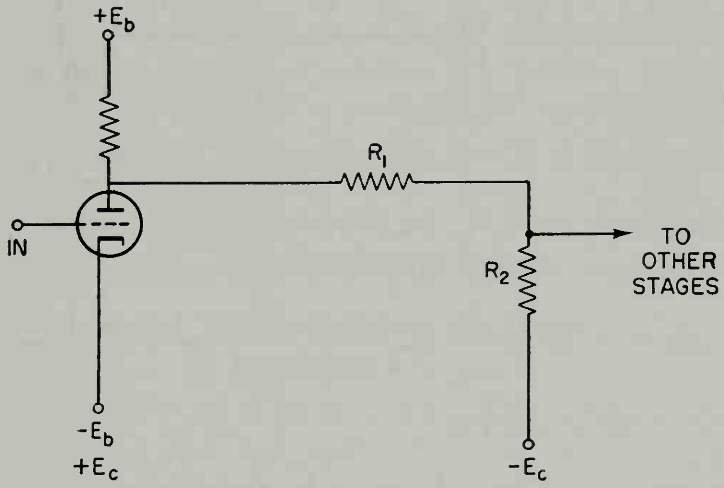

FIG. 5.2. Operation of d-c amplifier stages from common power supplies.

Drift in D-c Amplifiers. The output voltage of an a-c amplifier is zero for zero input voltage (except for the effects of various noise sources). The output level of a d-c amplifier, on the other hand, is sensitive to voltage-level changes throughout the amplifier and tends to change or *drift* with time. Drift will result in an *unbalance* voltage appearing at the amplifier output terminals in addition to the correct output voltage.

In computer applications, this unbalance leads to errors in the computation. Such errors may be especially serious if the unbalance voltage is integrated by an integrator.

The amount of unbalance due to drift is conveniently measured and described by the value E of the input d-c voltage needed to cancel the unbalance or to bring the output voltage back to the desired reference level. Figure 5.3 shows a d-c amplifier with three stages of gains a_1, a_2, and a_3 and a feedback ratio β. In the presence of an input voltage E and drift voltages e_1, e_2, e_3, e_4, the output voltage

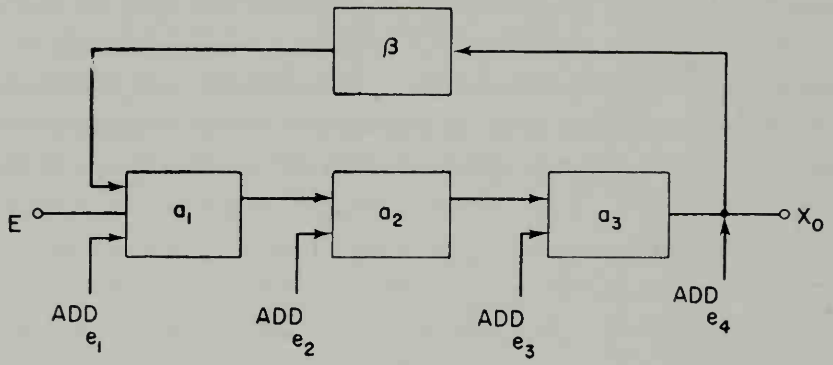

FIG. 5.3. Effects of drift in a multistage d-c amplifier with feedback.

$$X_o = \frac{a_1a_2a_3E + a_1a_2a_3e_1 + a_2a_3e_2 + a_3e_3 + e_4}{1 - a_1a_2a_3\beta} \tag{5.1}$$

will be brought back to zero if

$$E = -\left(e_1 + \frac{1}{a_1} e_2 + \frac{1}{a_1a_2} e_3 + \frac{1}{a_1a_2a_3} e_4\right) \tag{5.2}$$

The value of E is seen to be independent of the feedback networks used. Equation (5.2) also shows that the effects of drift are most serious in the earlier stages of a d-c amplifier, since the resulting unbalances will be amplified by succeeding stages.

The reduction of d-c amplifier drift is one of the most difficult design problems in the development of d-c analog computers and will be discussed in detail later in this chapter. The following outline describes the principal sources of drift together with some possible means of minimizing their effects. A diagram like Fig. 5.3 is usually helpful for analyzing the relative effects of various sources of drift in a d-c amplifier.

1. *Drift due to changes in plate and bias supply voltages.* Drift due to these causes may be reduced by employing well-regulated power supplies and possibly some of the compensating circuits described in Sec. 5.3.

2. *Drift due to changes in filament voltages.* This effect is particularly serious in the input stage of an amplifier. [One may effectively say that filament voltage changes contribute to the value of e_1 in Eq. (5.2)!] For most tubes (diodes, triodes, or pentodes) with oxide-coated unipotential cathodes,[1] a 20 per cent change in filament voltage is equivalent to a grid voltage change of 0.2 volt. The effects of filament voltage changes should be counteracted at least in the amplifier input stage by means of special compensating circuits (see Sec. 5.3) as well as through the use of regulated filament supplies.
3. *Drift due to changes in resistance values.* This drift can be minimized by using metalized or special wire-wound resistors of known quality. All critical resistors should be operated well below their power rating in order to reduce resistance changes due to heating. For best accuracy, the ambient temperature of the entire computer may be kept constant by means of a suitable air-conditioning system.
4. *Drift due to changes in vacuum-tube characteristics.* These changes are mostly due to changes in the cathode emission. The use of compensating circuits (Sec. 5.3) as well as *aging* of the vacuum tubes to be used will be helpful in this respect.

The actual amount of the unbalance voltages due to the various drift sources will depend on the actual circuits used in each case.

The Effects of Grid Current in D-c Amplifiers. Grid currents, or currents flowing from or into the grid, will cause voltage drops across the input resistors of computing amplifiers and thus lead to errors (distortion) in the machine variables. Even when the input signal is zero, grid current may cause an unbalance voltage which can often be balanced out by means of the balancing control. Changes in this zero-signal grid current constitute an additional source of drift. Since, for instance, a grid current of only 0.01 μa flowing in an input resistor of 1 megohm will give rise to an error voltage of 0.01 volt, it is seen that grid currents may lead to serious computing errors when their effects are amplified and integrated.

Figure 5.4 shows the variation of grid current i_g with grid voltage e_g for a typical triode operated so that the plate current i_p is constant.[1]

The current flowing from the cathode to the grid of the vacuum tube will be *negative* for sufficiently large negative values of the grid voltage e_g because positive ions of the residual gas in the tube are attracted to the negative grid. In some tubes, the latter may even emit electrons if some of the oxide coating from the cathode should be deposited on the grid. The amount of the negative grid current depends chiefly on the amount of residual gas in the tube and varies from tube to tube. The maximum allowable negative grid current of one section of a 6SL7 triode is given in the JAN specifications as 1 μa, but selected tubes may have negative grid currents as low as 0.02 μa. The JAN specifications for the 6AK5 are 0.1 μa and for the 6SU7, 0.02 μa. It may be helpful to operate the tubes at low plate, screen, and filament voltages in order to minimize ionization; the grid currents of specially evacuated and insulated electrometer tubes may be kept as low as 10^{-10} μa throughout their operating range in this manner.

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[1] *Ibid.*

As the value of the negative grid voltage is decreased, the grid current becomes *positive* owing to electrons flowing to the grid from the cathode.

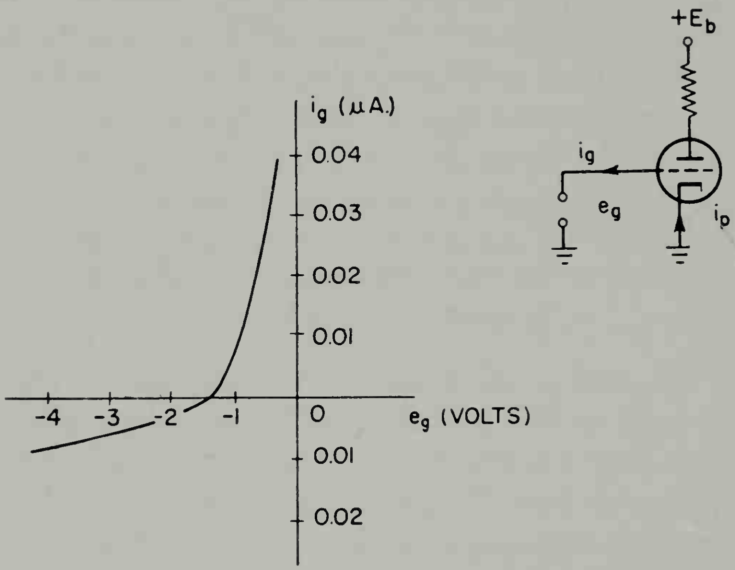


Fig. 5.4. Grid current vs. grid voltage for constant plate current in a triode.

This positive grid current tends to increase sharply as the grid voltage approaches zero (see Fig. 5.4) and will be larger for lower plate voltages. The rise in positive grid current determines the limits of the operating range with respect to bias and plate voltage or screen voltage in the case of pentodes.

Practical Balancing Circuits. In order to minimize computing errors arising from unbalances due to drift, *computer amplifiers should be balanced at least every half hour between computer runs and preferably before each run.*

Figure 5.5 shows a circuit suitable for switching a *summing amplifier* into the BALANCE condition which permits convenient balancing without any necessity for disconnecting the amplifier input and output terminals from the computer setup. The upper section of the *balancing switch* or push button serves to disconnect the amplifier input grid from the computing networks for balancing and to ground this grid through a resistance R_b equal to the resistance of the input resistors in parallel. The resistance between the input grid and ground is then approximately the same in the COMPUTE and BALANCE conditions; for during computation, either each input terminal is connected to a low-impedance voltage source (a preceding potentiometer or amplifier) or it is grounded directly through a closed-circuit jack as shown. Such an arrangement ensures that the zero-signal grid currents of the input stage are approximately equal during balancing and computing. The degenerative feedback through R_o also remains operative; while the feedback decreases the sensitivity of the balance indication, it is necessary to keep high-gain amplifiers and especially regenerative amplifiers stable during the balancing procedure.

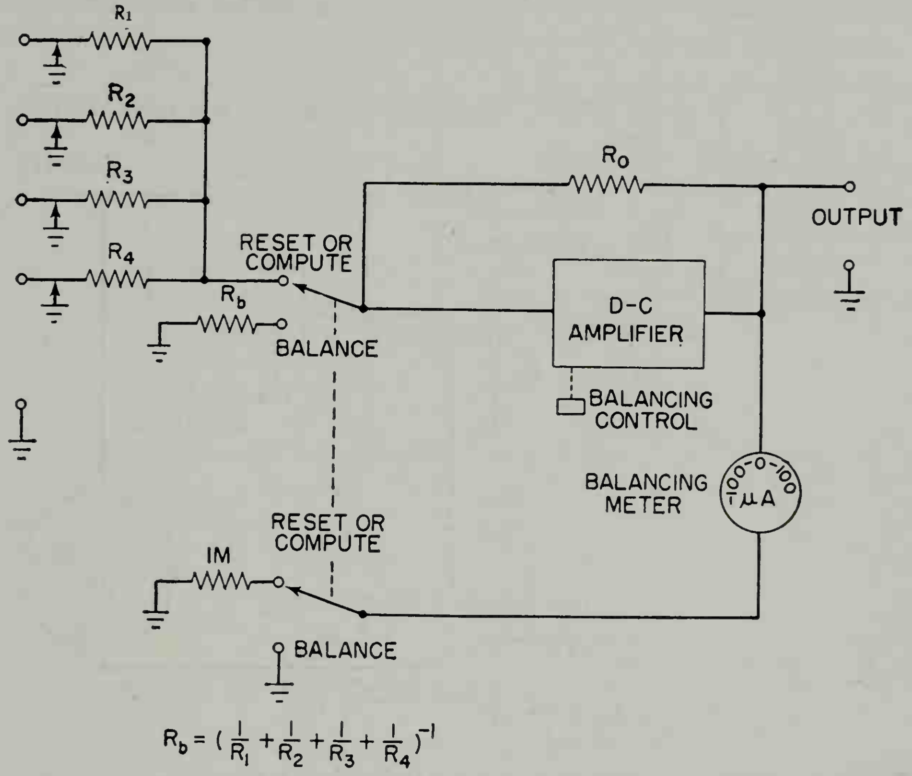
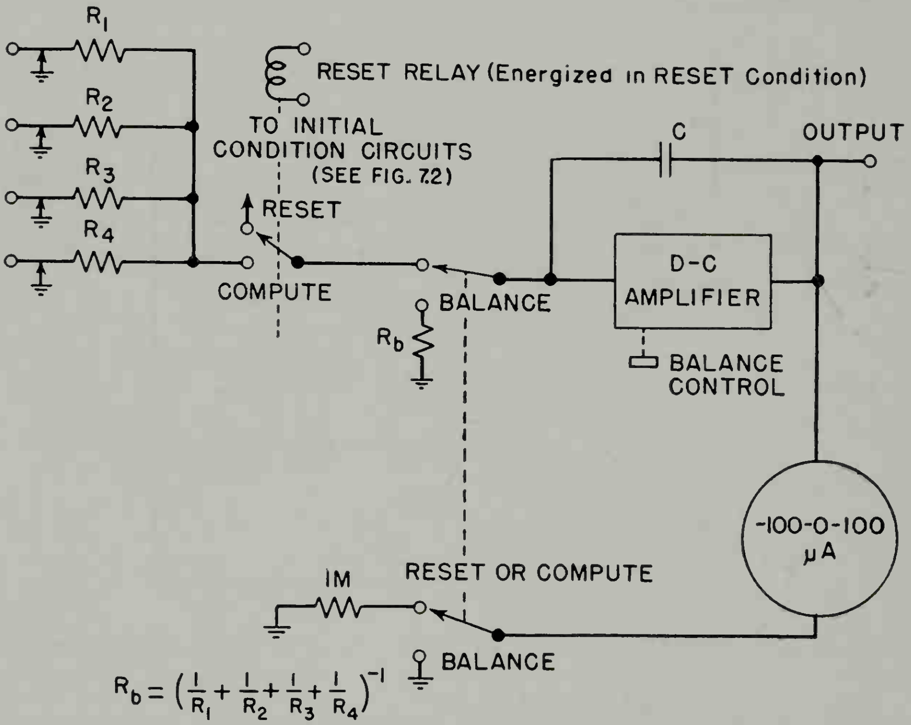


FIG. 5.5. Switching arrangement for balancing a summing amplifier.

The lower section of the balancing switch shorts a resistor in series with a microammeter which can then be used to read the unbalance voltage of the amplifier output. In the RESET and COMPUTE conditions, the same meter serves as a voltmeter which permits one to read the amplifier output voltage continuously.

The balancing circuit for a *summing integrator* shown in Fig. 5.6 is essentially identical with that for the summing amplifier. During the balancing procedure, the integrating feedback network comprising the capacitance C and the resistance R_b is operative, and the integrator will tend to integrate the unbalance voltage. This results in a very sensitive balance indication if a sufficient time interval is allowed after each balancing adjustment. The relay switching the integrator into the COMPUTE and RESET conditions is indicated in Fig. 5.6 (no HOLD circuits are shown). For balancing, the computer is placed in the RESET condition, and each amplifier is balanced in turn. The computer is then still in the RESET condition, ready for computation.

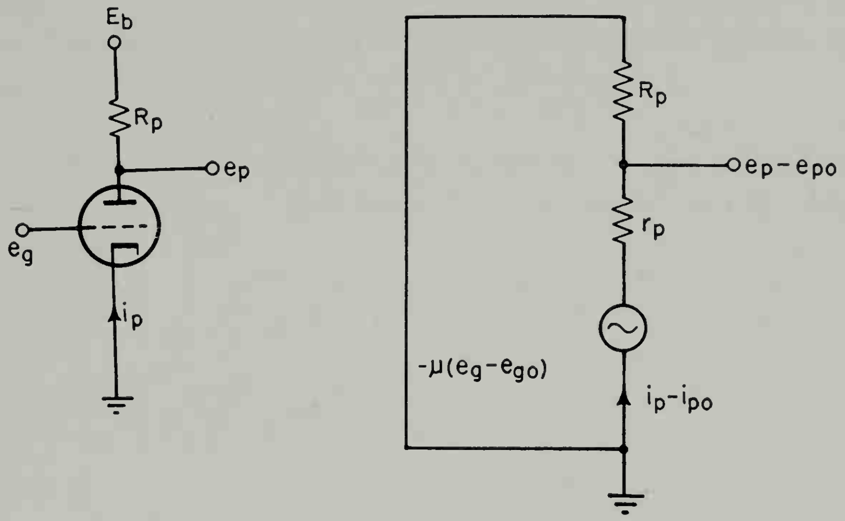
Many variations of the balancing circuits shown in Figs. 5.5 and 5.6 are possible. The sensitivity of the balancing meter may be adjustable by means of a range switch. The balancing switches may be replaced by a relay system which permits one to place all d-c amplifiers in the computer into the balance condition by means of a single switch. It is even possible to adjust each amplifier balance control by means of a special servomechanism actuated by the respective unbalance voltage. Generally speaking, such expensive refinements have been rendered unnecessary by the development of the *automatic balancing* circuits described in Sec. 5.6.



R_1
R_2
R_3
R_4
RESET RELAY (Energized in RESET Condition)
TO INITIAL CONDITION CIRCUITS (SEE FIG. 7.2)
RESET
COMPUTE
C
OUTPUT
BALANCE
R_b
D-C AMPLIFIER
BALANCE CONTROL
-100-0-100 μA
1M
RESET OR COMPUTE
BALANCE
$R_b = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)^{-1}$

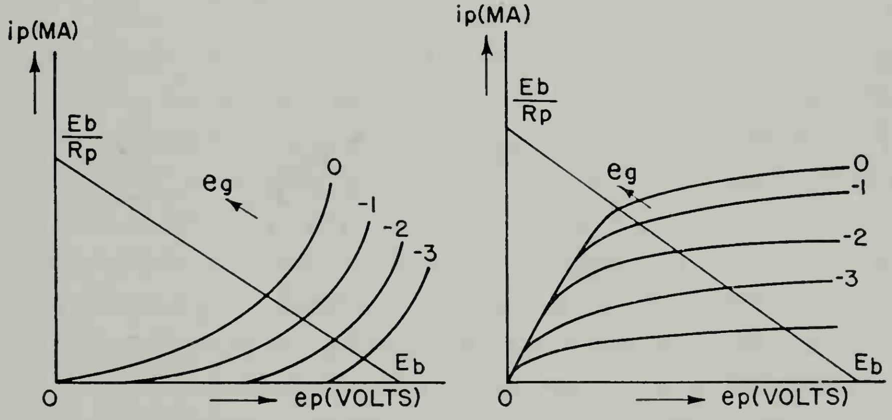
FIG. 5.6. Switching arrangement for balancing a summing integrator.

5.2. D-c Amplifier Design. *Design Procedure.* Ordinarily, the d-c amplifier designer will begin with a rough idea of the types of circuits and the number of stages needed to achieve a specific purpose. The actual design of the amplifier then proceeds stage by stage, usually beginning with the output stage; the values of the output voltage and current will, in general, have been previously specified. After the operating potentials for each tube have been chosen, the resistive networks needed to obtain the correct operating potentials from the given voltage sources must be designed. Finally, the entire design is gone over so as to effect necessary compromises between the characteristics of different stages and also to ensure the high-frequency stability of the amplifier. The amplifier may then be built and tested.



(*a*) (*b*)

FIG. 5.7. Basic vacuum-tube amplifier (*a*) and linear equivalent circuit (*b*).



(*a*) (*b*)

FIG. 5.8. Plate current and plate voltage characteristics of a triode (*a*) and of a pentode (*b*).

The operation of a voltage amplifier circuit of the type shown in Fig. 5.7*a* may be conveniently studied by means of the published plate voltage-current characteristics of the particular tube used. Figure 5.8 shows sets of such characteristics for a typical triode and a typical pentode. The *load lines* drawn in Fig. 5.8*a* and *b* for a particular value of the plate load resistor R_p serve to show the variation of the plate voltage e_p as a result of the voltage drop i_pR_p in the plate load resistor.

From such a set of characteristics, one can read the plate voltage e_p and the plate current i_p for all values of the grid voltage e_g for a given tube and load resistor combination. The technique of designing amplifiers from published characteristics is described in detail in the standard textbooks on electronics.[1]

In the vicinity of any *operating point* given by the values

$$e_p = e_{p0} \qquad i_p = i_{p0} \qquad e_g = e_{g0}$$

corresponding to zero input signal, the characteristics may be considered as straight lines, in which case it follows that

$$\left.\begin{aligned} \frac{i_p - i_{p0}}{e_g - e_{g0}} &= \frac{\mu}{r_p + R_p} = \frac{g_m}{1 + \dfrac{R_p}{r_p}} \\ \frac{e_p - e_{p0}}{e_g - e_{g0}} &= \frac{-\mu R_p}{r_p + R_p} = -g_m \frac{R_p r_p}{R_p + r_p} \end{aligned}\right\} \tag{5.3}$$

where the three derivatives

$$r_p = \left.\frac{\partial e_p}{\partial i_p}\right]_{e_g} \qquad g_m = \left.\frac{\partial i_p}{\partial e_g}\right]_{e_p} \qquad \mu = \left.\frac{\partial e_p}{\partial e_g}\right]_{i_p} = r_p g_m \tag{5.4}$$

are known as the *plate resistance*, *transconductance*, and *amplification factor* of the tube, respectively. The linear *equivalent circuit* shown in Fig. 5.7*b* is a very convenient aid for deriving approximate linear relations like (5.3) between voltage and current *changes* in the corresponding vacuum-tube circuit of Fig. 5.7*a*. If the voltage and current changes take place sufficiently rapidly, the effects of tube and circuit capacitances must be taken into consideration in the manner shown in Sec. 5.4.

The *choice of the operating point* cannot be made from a consideration of the linear equations (5.3) alone, since the values of r_p, μ, and g_m change with the operating point. Instead, the characteristics shown in Fig. 5.8*a* and *b* must be consulted. The operating point for a voltage amplifier will be selected so that the gain is as high as possible for the particular circuit used, while the gain changes due to distortion still remain below a certain tolerable value, usually about 1 to 2 per cent. For an output stage, both the voltage and current gain must be sufficiently high, while the distortion is kept within tolerable limits.

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[1] *The Radiotron Designer's Handbook*, 3d ed., edited by F. Langford Smith, Radio Printing Press Pty. Ltd., Sydney, Australia. Reproduced and distributed in the United States by RCA Manufacturing Company, Inc., 1941; F. E. Terman, *Radio Engineers' Handbook*, McGraw-Hill, New York, 1943; MIT Electrical Engineering Staff, *Applied Electronics*, The Technology Press, 1943; D. G. Fink, *Engineering Electronics*, McGraw-Hill, New York, 1938; G. E. Valley and H. Wallman, *Vacuum Tube Amplifiers*, MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948.

Particularly in the case of pentodes a wide variety of operating conditions is possible, since the various grid voltages can all be varied. The references given in the preceding footnote should be consulted for more detailed discussions of these questions; an exhaustive treatment of amplifier design is entirely beyond the scope of this book.

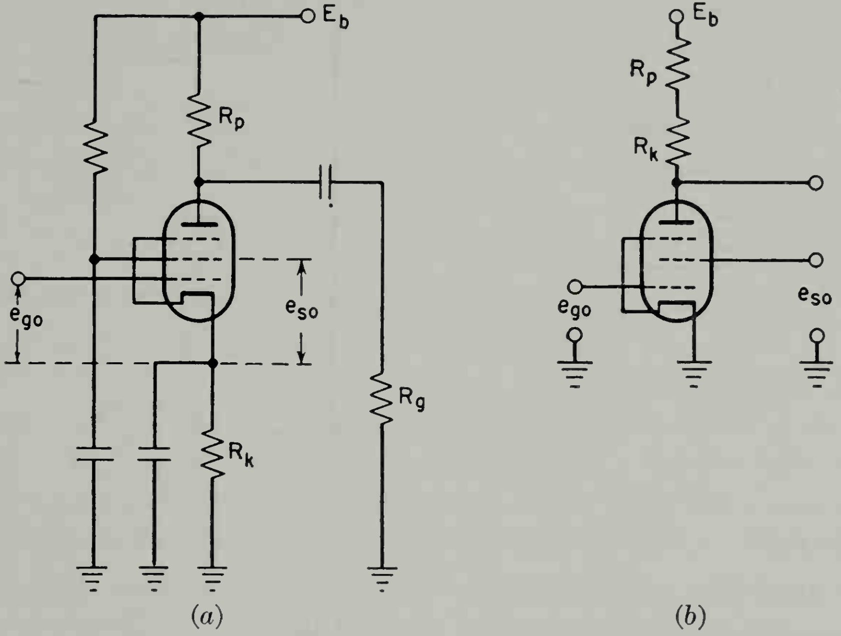
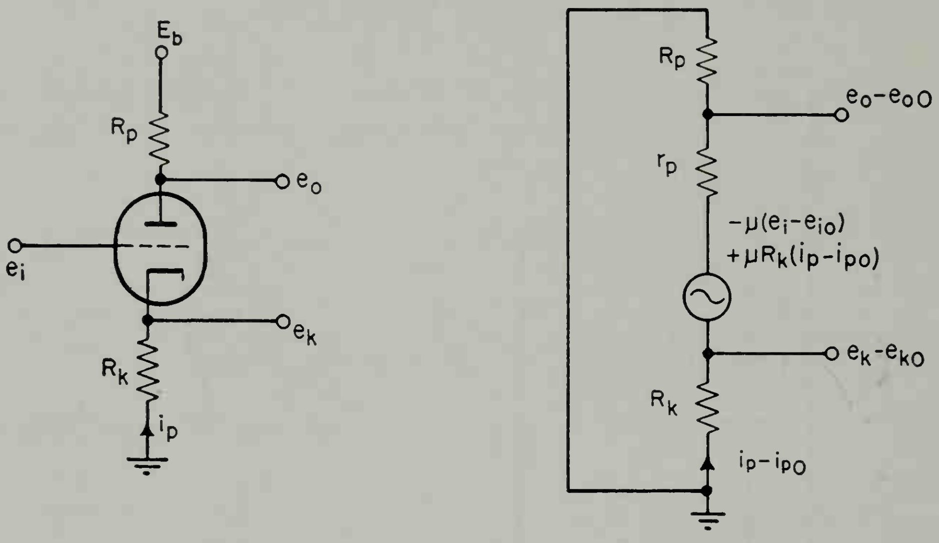


FIG. 5.9. Development of a d-c amplifier stage from a resistance-coupled a-c amplifier. (In the case of triodes, disregard the screen- and suppressor-grid circuits.)

The Use of Resistance-coupled Amplifier Tables. In view of the labor involved in designing d-c amplifiers directly from the tube characteristics, it may be expedient to adopt vacuum-tube operating conditions similar to those given in *standard resistance-coupled amplifier tables.* Such resistance-coupled amplifier data are given in most tube handbooks and usually refer to an a-c voltage amplifier stage of the type shown in Fig. 5.9*a*, having a by-passed cathode resistor R_k. The tabulated data may be used to design an "equivalent" d-c amplifier shown in Fig. 5.9*b*. This d-c amplifier will have approximately the same operating point as the original a-c amplifier if the resistance of the load connected to it is high; this is usually the case for voltage amplifier stages. The equivalent amplifier has fixed bias and, in the case of pentodes, a fixed screen voltage. The gain will, in general, be slightly higher than that given in the tables, since the plate load resistance is higher and also because the resistance of the load connected to the d-c amplifier stage often will be higher than the resistance R_g specified in the tables.

In a-c amplifiers, the value of the plate load resistance R_p is often limited by the high-frequency response required of the stage. In d-c amplifiers, it is often permissible to use rather large values of R_p and correspondingly small plate currents in order to obtain a large voltage gain. In the case of pentodes, maximum stage gain can be obtained by using low screen voltages as well as low plate voltages.[1] The lowest permissible values of plate and screen voltages will be determined by the amount of positive grid current which can be tolerated.



(a) (b)

FIG. 5.10. Amplifier stage with cathode bias and equivalent circuit.

The Effects of Cathode Bias Resistors and Screen Series Resistors. If the grid bias for the operation of an amplifier tube is obtained by means of a cathode resistor in the manner shown in Fig. 5.10*a*, the net voltage between grid and cathode becomes

$$e_g = e_i - i_pR_k$$

The current and voltage changes may again be derived from the equivalent linear circuit as shown in Fig. 5.10*b*.[2] The grid-to-plate gain of the stage is

$$\frac{e_o - e_{o0}}{e_i - e_{i0}} = \frac{-\mu R_p}{R_p + [r_p + (\mu + 1)R_k]} \tag{5.5}$$

where e_i and e_{i0} are the input and reference input voltages, respectively, and e_o and e_{o0} are the output and reference output voltages, respectively. The grid-to-cathode gain is found to be

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[1] Gray, *op. cit.;* Volkers, W. K., Ultra-high Gain Direct-coupled Amplifier Circuits, lecture presented at the IRE National Convention, New York, 1950.

[2] This circuit was used in Chap. 4 to describe integrator performance.

$$\frac{e_k - e_{k0}}{e_i - e_{i0}} = \frac{\mu R_k}{R_p + [r_p + (1 + \mu)R_k]} \tag{5.6}$$

where e_k and e_{k0} are the cathode voltage and its reference voltage, respectively. All voltages are measured with respect to the point shown grounded in Fig. 5.10.

The values of r_p, g_m, and μ to be used in Eqs. (5.5) and (5.6) are again determined by the effective operating point. The latter may be found from the tube characteristics through successive approximations of the plate current i_p.[1]

A comparison of Eqs. (5.3) and (5.6) shows that a tube in the circuit of Fig. 5.10*a* acts like a tube of "equivalent dynamic plate resistance"

$$Z = r_p + (1 + \mu)R_k \tag{5.7}$$

connected in the simpler circuit of Fig. 5.7*a*. It is also seen that the inverse feedback due to the cathode resistor has decreased the gain of the stage. A similar effect is produced by a screen dropping resistor used with a pentode; the corresponding analysis is quite complicated.[2] Because of the loss in gain, the use of cathode and screen dropping resistors to obtain correct electrode voltages is often avoided in favor of fixed voltage sources. On the other hand, the degenerative effect, especially of the cathode dropping resistor, tends to reduce the effects of amplifier gain changes and distortion.

It should be noted that in push-pull amplifier circuits like the one shown in Fig. 5.11, the degenerative effect of the cathode and screen resistors is canceled or at least reduced because the plate current changes in the two tubes oppose each other. In such circuits, the full stage gain given by Eq. (5.3) can be realized in spite of the cathode and screen dropping resistors.

The Effect of Load Resistance. Each computer amplifier stage will, in general, be connected to a resistive *load* of resistance R_L, say; the load may be returned to ground or to some fixed voltage source of voltage E_c with respect to ground as shown in Fig. 5.12.

The effect of the load on the amplifier performance is best analyzed by drawing a new load line across the characteristics of the amplifier tube. Such a load line is shown in Fig. 5.12*b* for a triode amplifier and expresses the relation between the plate voltage e_p and the plate current i_p as

[1] Gray, *op. cit.*

[2] *Ibid.*

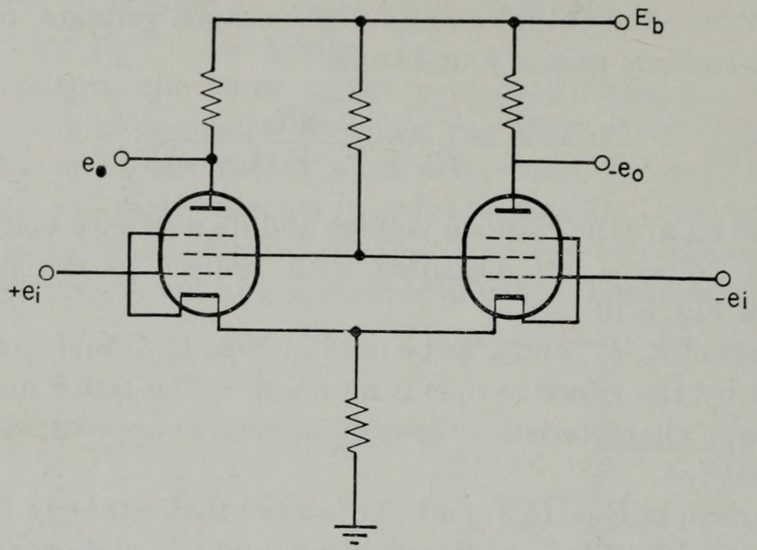
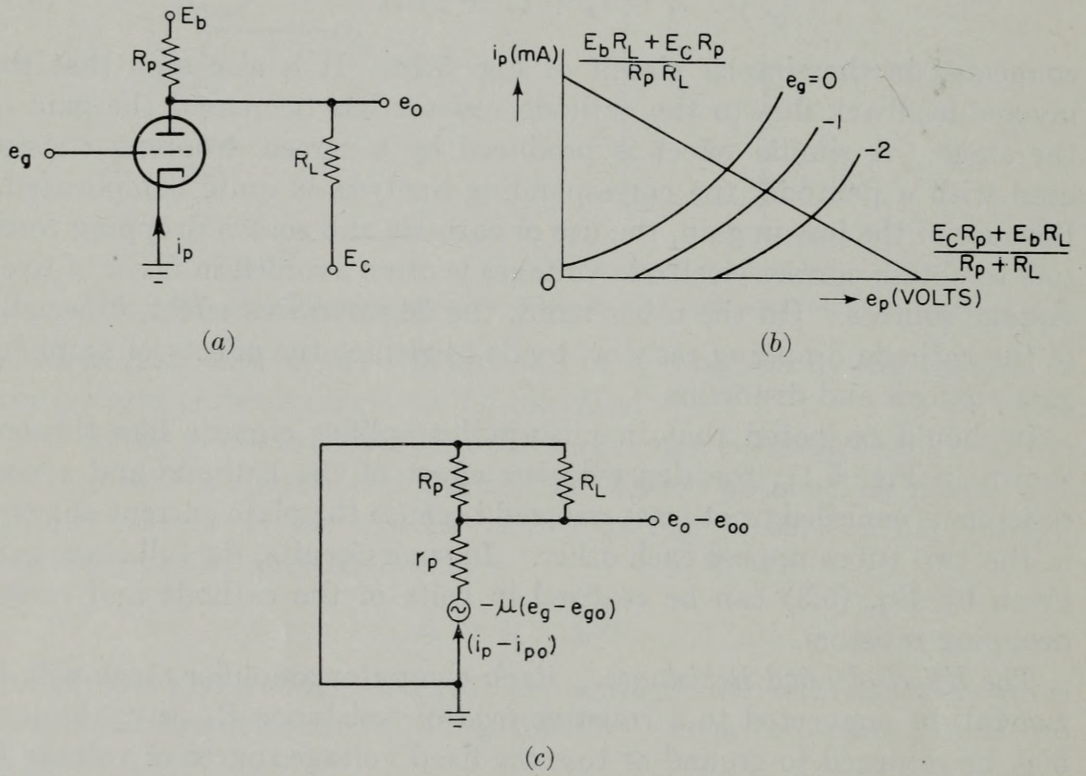


FIG. 5.11. A push-pull amplifier circuit.



(a)

(b)

(c)

FIG. 5.12. Illustrating the analysis of a simple amplifier stage with a load of resistance R_L.

determined by the values of R_p, R_L, E_b, and E_c. The value of R_L for many voltage amplifier stages is so large that the effect of the load is negligible, so that the load lines of Fig. 5.8 can still be used. This is not usually true in the case of output stages. When the operating point on some desirable load line has been chosen, it is possible again to use the linear equivalent circuit for the vacuum tube to derive the voltage and current changes. Figure 5.12 shows the linear equivalent circuit for Fig. 5.12. The stage gain is now easily derived by substituting the *total load resistance*

$$R_p' = \frac{R_p R_L}{R_p + R_L} \tag{5.8}$$

for the plate load resistance in Eqs. (5.3). Similar equivalent circuits may be used to derive the gain of other loaded amplifier stages.

Interstage Coupling. The design equations for the coupling resistors used to establish the correct d-c reference level at the grid of the succeeding stage may be derived directly from the circuit of Fig. 5.13. Suppose that the positive and negative power-supply voltages E_b and E_c as well as the reference voltage e_{o0} at the plate have been chosen and that it is desired to establish a reference voltage level E_{o0} at the grid of the succeeding tube. Then the required values of the resistors R_1 and R_2 making up the interstage coupling network may be derived from the relations

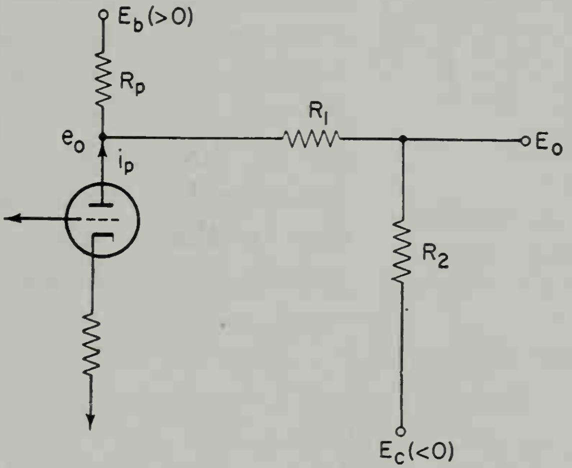


FIG. 5.13. Design of interstage networks.

$$\left.\begin{aligned} R_1 &= \frac{e_{o0} - E_{o0}}{E_b - e_{o0} - i_p R_p} R_p \\ R_2 &= \frac{E_{o0} - E_c}{E_b - e_{o0} - i_p R_p} R_p \end{aligned}\right\} \tag{5.9}$$

The stage gain $(e_o - e_{o0})/(e_i - e_{i0})$ may be derived as shown in the preceding paragraphs. The combined gain of the amplifier stage and coupling network is

$$\frac{E_o - E_{o0}}{e_i - e_{i0}} = \frac{e_o - e_{o0}}{e_i - e_{i0}} \frac{R_2}{R_1 + R_2} \tag{5.10}$$

It may be seen that the loss of gain due to the interstage resistive network can be reduced by increasing E_c and, therefore, R_2.

5.3. Special D-c Amplifier Circuits. *Some Special Applications of Gas-filled and Vacuum Tubes.* It is often desirable in d-c amplifier circuit design to have circuit elements whose *incremental resistance* (*i.e.*, the ratio between corresponding voltage and current changes) is different from their *d-c resistance* (corresponding to the ratio of the total voltage and current).

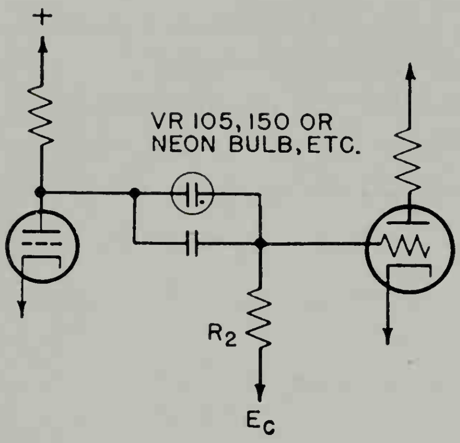


FIG. 5.14. Use of a gas-discharge tube in an interstage coupling network.

Figure 5.14 shows the use of a voltage regulator tube or simply a neon bulb as a circuit element of *low incremental resistance* in an interstage network. The gas discharge tube will tend to maintain a constant voltage across its electrodes. Accordingly, in the circuit[1] shown, the gas tube permits one to bring the voltage at the grid of the second stage to the desired level without any loss in gain due to the interstage networks. The gas discharge tube acts like an infinitely small resistance with respect to voltage changes, while still maintaining a finite d-c voltage across itself. The capacitor shown in Fig. 5.14 serves to by-pass any noise which might be generated by the gas discharge tube.

Some examples of the use of the dynamic plate resistance

$$Z = r_p + (\mu + 1)R_k$$

of a vacuum tube to obtain a *high incremental resistance* have already been shown in Sec. 4.4. Figure 5.15 shows four examples of the use of vacuum tubes as circuit elements of high incremental resistance in d-c amplifier circuits. Figure 5.15*a* and *b* illustrates the use of a vacuum tube for simulating a large plate load resistance in an amplifier and a large cathode resistance in a cathode follower, respectively. In either case, a large load resistance is simulated whereas the d-c operating voltages on the electrodes of the amplifying tube can still be reasonably high. Arrangements of this type often result in better linearity than is possible with circuits using resistors. Figure 5.15*c* shows the use of a tube as the common cathode resistor in a differential amplifier. In this circuit, the high effective cathode resistance helps to make the output voltages of the two halves of the amplifier more nearly equal and opposite (see page 178).

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[1] Miller, S. E., Sensitive D-c Amplifier with A-c Operation, *Electronics*, November, 1941.

Figure 5.15*d* shows how the dynamic plate resistance of a triode may be used to decrease the attenuation of an interstage coupling network. In this circuit the tube acts like a large incremental resistance while still maintaining the correct operating voltage at the grid of the succeeding stage.

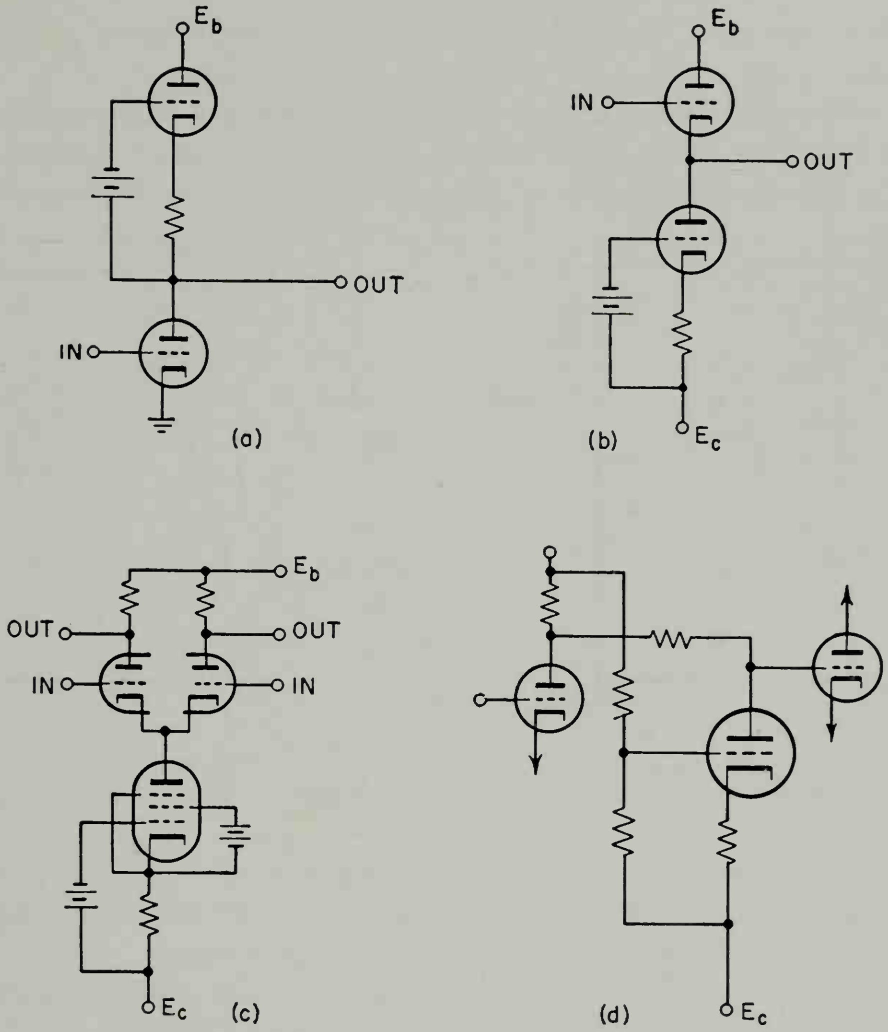

FIG. 5.15. Examples of the use of the dynamic plate resistance of a vacuum tube to obtain high incremental resistance.

Compensation for Supply Voltage Changes: Bridge-balanced D-c Amplifiers. The two vacuum tubes used in the *series-compensated* amplifier[1] shown in Fig. 5.16*a* form two arms of a bridge. If the output voltage is taken off at the terminals indicated in the diagram, the zero-signal output voltage can be balanced to zero by adjusting one of the cathode resistors as shown and will then be, to a first approximation at least, independent

[1] Artzt, Maurice, Survey of D-c Amplifiers, *Electronics*, August, 1945.

of plate and filament supply voltage variations. If a load of resistance R_L is connected between the output terminals, the voltage gain is given by

$$\frac{e_o - e_{o0}}{e_i - e_{i0}} = \frac{-\mu R_L}{(\mu + 1)R_k + r_p + 2R_L + R} \tag{5.11}$$

assuming that two identical tubes of amplification factor μ and plate resistance r_p are used.

The compensation for cathode emission changes in such arrangements will be best if the two tubes are as closely identical as possible. Two tubes in one envelope may be used if the cathode-to-filament voltage rating of the tube is not exceeded. It may still be difficult to match the operating conditions for the two halves of the tube accurately, since the two cathode-to-filament voltages cannot be made equal.

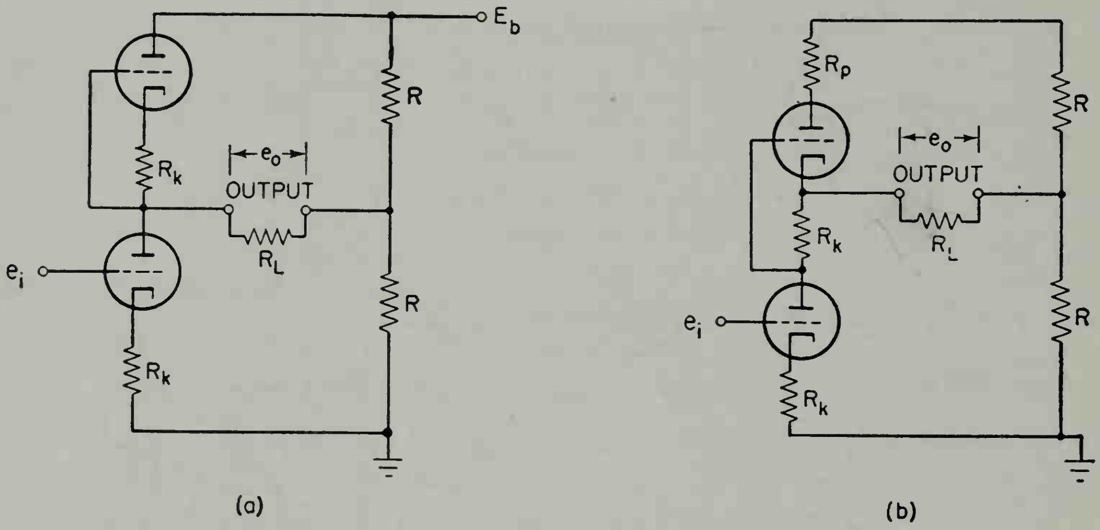

FIG. 5.16. Bridge-balanced (series-compensated) d-c amplifier stages.

The gain given by Eq. (5.11) will assume the value $\mu/2$ if R_L is infinitely large. The value of $\mu/2$ may be as high as 4,000 for pentodes. This high gain is predicated on the high effective load resistance afforded by the dynamic plate resistance of the tube V_2. Equation (5.11) shows that the gain will decrease substantially if V_2 must be by-passed by a finite load resistance R_L. In such cases, the gain of the circuit shown in Fig. 5.16*b* is, if $R_p = 2R_k$,

$$\frac{e_o - e_{o0}}{e_i - e_{i0}} = \frac{-\mu R_L}{r_p + 2R_k + 2R_L + R} \tag{5.12}$$

which is somewhat higher than that of the circuit of Fig. 5.16*a* for the same circuit constants.

A new type of bridge-balanced d-c amplifier circuit,[1] shown in Fig. 5.17*a*, has all the advantages of series compensation, and, at the same time, its output voltage has the same d-c reference level as the input voltage. In Fig. 5.17*a*, this has been made possible through the use of a single floating power supply. This circuit is particularly suitable for small, portable feedback amplifiers for battery operation, in which case the floating power supply may not be objectionable. The gain of the circuit of Fig. 5.17*a* is the same as that given by Eq. (5.11) for the circuit of Fig. 5.16*a*.

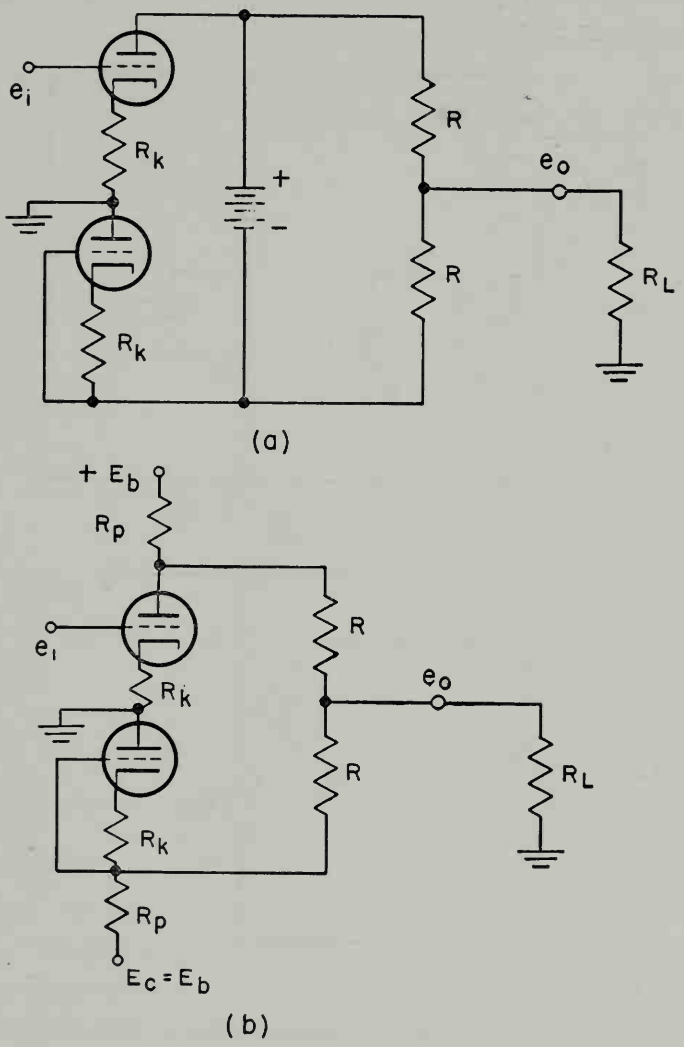

FIG. 5.17. A new type of bridge-balanced d-c amplifier circuit.

If it is desirable to cascade several stages using common power supplies, the circuit of Fig. 5.17*b* may be used.

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[1] Korn, G. A., and T. M. Korn, Bridge-balanced D-c Amplifier, *Radio and Television News*, Engineering Edition, January, 1950.

Figure 5.18 shows two low-impedance driver circuits (current amplifiers) using series compensation. The gain of each circuit is

$$\frac{e_o - e_{o0}}{e_i - e_{i0}} = \frac{\pm \mu R_L}{r_p + (\mu + 1)R_k + (\mu + 2)R_L + R} \tag{5.13}$$

where the positive sign applies to the circuit of Fig. 5.18*a* and the negative sign to the circuit of Fig. 5.18*b*.

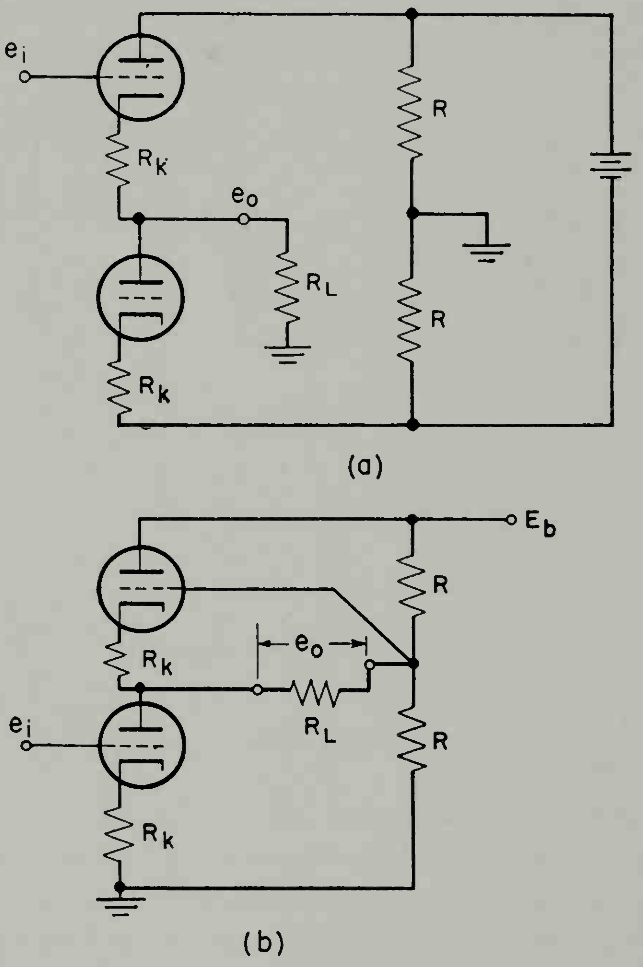

Fig. 5.18. Bridge-balanced low-impedance driver circuits.

The output reference level of series-compensated amplifier stages is particularly insensitive to changes in the plate supply voltage. It is possible to turn the plate supply on and off without changing the output reference level. As a matter of fact, in certain applications, bridge-balanced amplifiers have been operated with alternating current on the plate without adverse effects.[1]

Special Circuits for the Compensation of Filament Voltage Changes. It has been mentioned in Sec. 5.1 that the d-c amplifier drift due to filament voltage changes is particularly serious, since filament voltage changes affect each amplifier stage exactly like input voltage changes. It is, therefore, customary to use some type of *compensating circuit* at least in the input stage of good-quality d-c amplifiers. The technique of *series compensation* has been described above; a number of other compensating circuits will be described in the following paragraphs.

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[1] Artzt, *op. cit.*

Figure 5.19*a* shows how the effects of filament voltage variations in an amplifier tube V_1 may be at least partially compensated by means of a diode V_2 having the same filament voltage as V_1. When the filament voltage changes, the diode plate current flowing through the cathode resistor R_k changes accordingly. The resulting bias change tends to compensate for the effects of the original filament voltage variation. Tubes with diodes using the same cathode as the triode or pentode amplifier tube V_1 are especially useful in this circuit, since the characteristics of the cathode circuits of the two tubes are more nearly identical in this case.[1] It is also possible to use the control grid of certain beam tetrodes, particularly of the type 6V6, instead of the diode plate; in this case, the screen grid functions as the control grid (see Fig. 5.19*b*). If the circuit constants are chosen correctly, quite effective compensation may be obtained, since the cathode of the diode and the cathode of the amplifier tube are actually identical in this circuit.

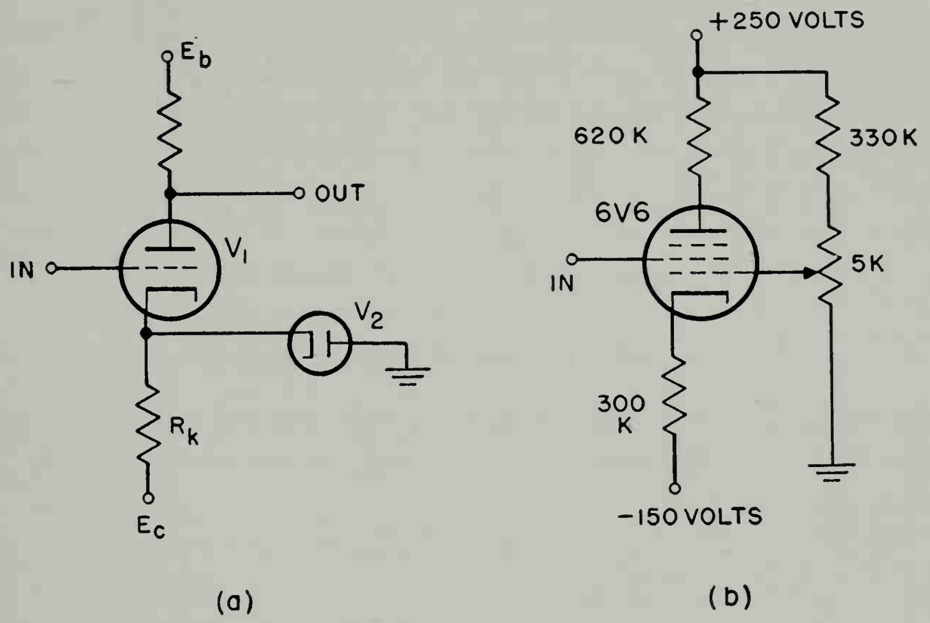

FIG. 5.19. Circuits for the compensation of the effects of filament voltage changes.

Probably the most commonly used circuits for compensation of the effects of filament voltage changes and cathode emission are the differential amplifiers treated below.

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[1] Gray, *op. cit.*

The Use of Differential Amplifiers. In the two circuits shown in Fig. 5.20*a* and *b*, each of the two tubes acts as a cathode follower driving the cathode of the other tube. The changes in the output voltages of these circuits may be obtained as functions of the corresponding changes in the input voltages through the use of linear equivalent circuits in the manner outlined in Sec. 5.2. Assuming that the two tubes are identical types in each case, the analysis shows that for the circuit of Fig. 5.20*a*,

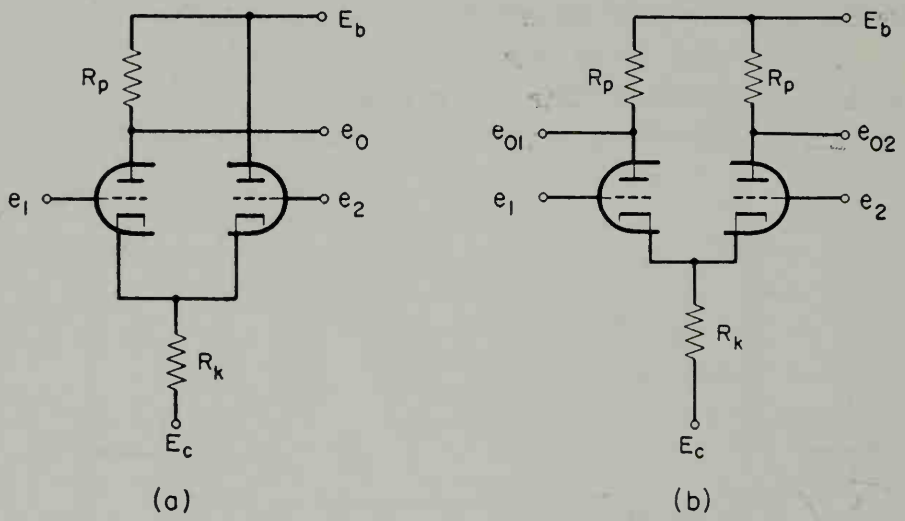

FIG. 5.20. Differential amplifier circuits.

$$(e_o - e_{o0}) = \frac{\mu}{1 + \dfrac{2r_p}{R_p} + \dfrac{r_p(r_p + R_p)}{R_pR_k(\mu + 1)}}(e_2 - e_{20}) - \frac{\mu + \dfrac{\mu}{\mu + 1}\dfrac{r_p}{R_k}}{1 + \dfrac{2r_p}{R_p} + \dfrac{r_p(r_p + R_p)}{R_pR_k(\mu + 1)}}(e_1 - e_{10}) \qquad (5.14)$$

and for the circuit shown in Fig. 5.20*b*,

$$\left.\begin{aligned}(e_{o1} - e_{o0}) &= \frac{\mu R_p}{(R_p + r_p)\left[2 + \dfrac{R_p + r_p}{(1 + \mu)R_k}\right]} \left\{(e_2 - e_{20}) - (e_1 - e_{10})\left[1 + \frac{R_p + r_p}{(\mu + 1)R_k}\right]\right\} \\ (e_{o2} - e_{o0}) &= \frac{\mu R_p}{(R_p + r_p)\left[2 + \dfrac{R_p + r_p}{(1 + \mu)R_k}\right]} \left\{(e_1 - e_{10}) - (e_2 - e_{20})\left[1 + \frac{R_p + r_p}{(\mu + 1)R_k}\right]\right\}\end{aligned}\right\} \qquad (5.15)$$

In each case, e_{10} and e_{20} are the reference levels of the input voltages e_1 and e_2, respectively. From Eqs. (5.14) and (5.15), it may be seen that the output voltages of these circuits are, for sufficiently high values of μ and R_k, approximately proportional to the differences of the two input voltage changes. For this reason, such circuits are referred to as *differential amplifiers.* The circuit of Fig. 5.20*b* permits output voltages of either sign to be taken off. The flexible properties of differential amplifiers of this type make them extremely useful in many circuit applications. Some of their many interesting properties are listed below:

1. *Compensation of the effects of power-supply voltage changes in the input circuits.* Since the output voltages of differential amplifiers are approximately proportional to the differences between the two input voltages, changes in the input voltages due to power-supply voltage variations common to both *input* circuits will tend to cancel out. In order to make the output of a differential amplifier as closely proportional to the difference between the two input voltages, it is advisable to use tubes having a high amplification factor μ and to use a high value of the cathode resistor R_k. The latter is made possible through the use of a large negative voltage for the cathode return. Otherwise, the plate currents may become too small, and the amplification factors of the tubes may decrease; it is also possible to replace R_k by the dynamic resistance of a tube in the manner shown in Fig. 5.15*c*. One might think that pentodes would be particularly useful as differential amplifier tubes because of the high value of μ. The value of r_p for pentodes is, however, so high that Eqs. (5.14) and (5.15) show that pentodes are *not* preferable to high-μ triodes in this application. Specially matched double triodes, like the 6SU7 or 5691, are most suitable and also have a low negative grid current rating.

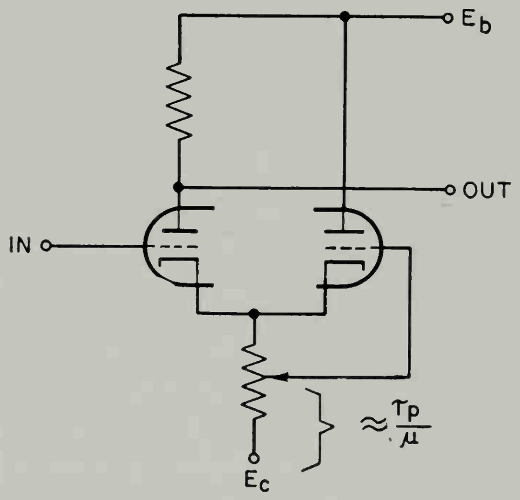

FIG. 5.21. Miller compensator.

2. *Compensation of the effects of cathode emission changes.* Changes in the cathode emission due to filament voltage changes and other causes are equivalent to voltage changes common to both input circuits of differential amplifiers. They are, therefore, at least partially canceled. Figure 5.21 shows a special circuit for canceling the effects of filament voltage changes more accurately (the so-called Miller compensator circuit).[1] It

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[1] Miller, *op. cit.*

may be shown that the resistance below the cathode resistor tap in Fig. 5.21 should be equal to r_p/μ. In order to make the compensation reasonably permanent, the tubes should be aged for a while and cycled several times through the extremes of heater voltage before the adjustment is made.[1]

The value of R_k is not critical but should be between two and eight times the value of r_p/μ. The adjustment is best made by observing the output voltage while changing the heater voltage.

3. *Compensation for the effects of changes in plate and bias supply voltages.* Figure 5.22 shows a differential amplifier arranged for compensation of the effects of changes in plate and bias voltages. If the same power supplies are used for all stages of an amplifier, the circuit of Fig. 5.22 may be arranged so as to serve as a compensating circuit for the entire amplifier. At the same time, the circuit still provides a measure of compensation for the effects of changes in its own cathode emission.

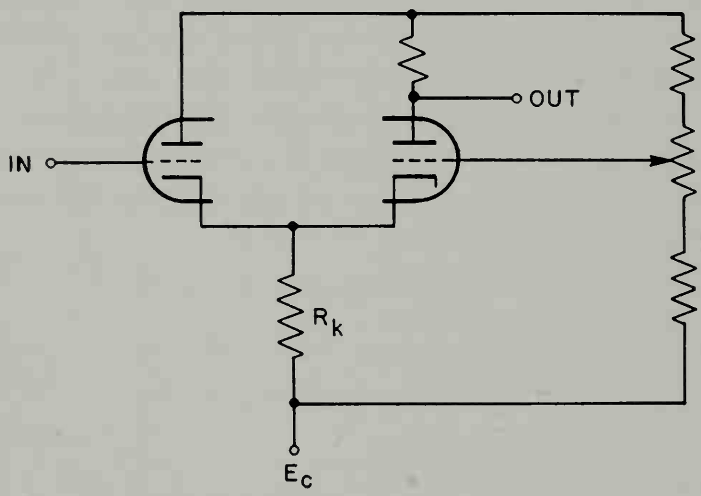

Fig. 5.22. Compensation for the effects of plate and bias supply voltage changes.

Generally speaking, the plate and bias supplies in d-c analog computers are well regulated for reasons other than the prevention of amplifier drift (see Sec. 7.4). Accordingly, circuits permitting compensation for plate voltage variations are not too frequently used in computer amplifiers.

4. Another significant advantage of differential amplifiers is that *they permit the output voltage to be taken off either with or without a sign reversal referred to the input voltage.* This allows great flexibility in various applications, particularly in circuits involving feedback around one or more amplifier stages.

5. A differential amplifier makes it possible to *mix two voltages by taking their difference.* In this manner, one may, for instance, introduce an adjustable balance voltage for setting the output reference level of a d-c

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[1] Gray, *op. cit.*

amplifier. Again, feedback voltages may be conveniently introduced at one of the two grids in the sense desired for either degeneration or regeneration.

6. In differential amplifiers, the plate currents of the two tubes are approximately equal and opposite, and for this reason, *cathode bias can be used in a differential stage without any corresponding loss in gain through degeneration.*

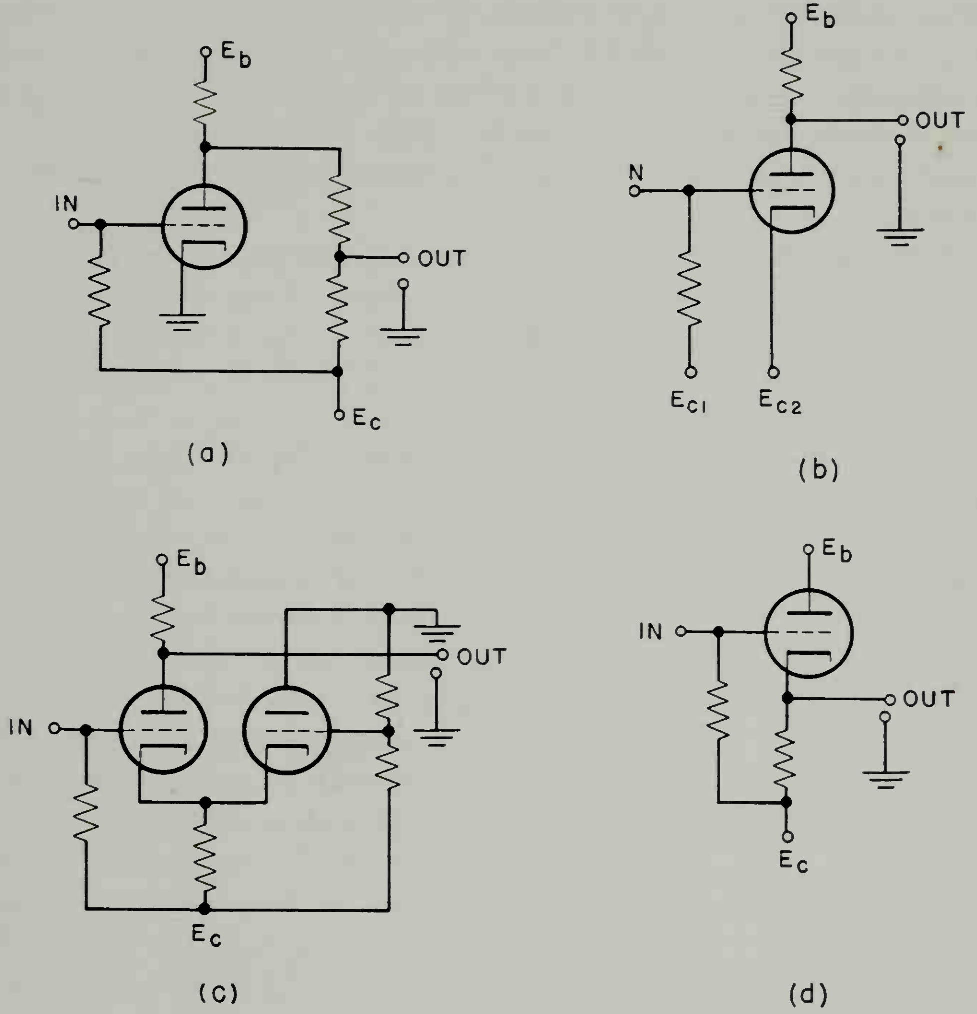


FIG. 5.23. Output circuits with zero d-c reference voltage with respect to ground.

7. *Push-pull computer amplifiers* may be arranged by cascading symmetrical differential amplifiers. Push-pull amplifiers permit one to obtain output voltages of either sign; they may thus be used to eliminate phase-inverting amplifiers in computer setups (see also Sec. 5.5).

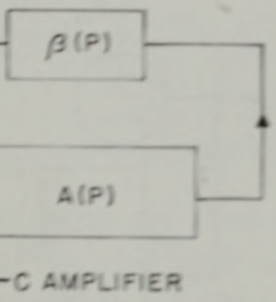
Several examples of d-c amplifier designs utilizing these properties of differential amplifiers are presented in Sec. 5.5.

Output Circuits. As mentioned in Sec. 5.1, the output reference level of d-c amplifiers used in computer applications must be the same as the input reference level, usually zero with respect to ground. Figure 5.23

a number of output circuits suitable for computer amplifiers. Fig-3 represents the already familiar arrangement using a bias voltage ring the reference voltage of the output terminal to ground level. uble with this arrangement is that, in an output stage, a reasonable t of current may have to flow through the load, so that a consider- nount of power might be wasted in the coupling network. In such the circuit of Fig. 5.23b is preferred. In this frequently used ment, the reference voltage of the output tube plate itself is t to ground level by the bias voltage E_{c2}. In this circuit, a sepa- as supply must be provided to supply grid bias for the output tube cathode resistor bias is used. This, in turn, would result in a e in gain together with an improvement in the linearity of the out- ge.

degeneration due to a cathode resistor may be avoided through the a differential amplifier output stage, as shown in Fig. 5.23c. , cathode-follower output, as shown in Fig. 5.23d, may be used od results; such an output stage will, however, not contribute to tage gain of the amplifier.

Stabilization of Feedback Amplifiers. *Introduction.* The oper- amplifiers used in d-c analog computers are special types of feedback amplifiers. Thus, the parallel-feedback types of operational amplifiers constitute feedback loops like that shown in Fig. 5.24.



4. Feedback loop by a parallel-feed- rational amplifier.

It is convenient to discuss the response of feedback amplifiers in terms of the responses to sinusoidal components of the applied voltages. It is possible to prove that *feedback amplifiers like that shown in Fig. 5.24 will break into uncontrolled oscillations if, for some rang of frequencies, the phase shift inherent in th loop transfer function Aβ deviates from* 180 *de e than* 180 *deg while the absolute value of Aβ is equal to or great ity.*[1]

vely speaking, the feedback will then be sufficiently regenerative he amplifier unstable. In order to design amplifier circuits whi

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may be derived from the general necessary and sufficient condition (given . W., *Network Analysis and Feedback Amplifiers*, Van Nostrand, New Yo at, for stability, all roots p_k of the characteristic equation must have posit s. Assuming that all circuit constants including the load and source imp e given, the characteristic equation of the feedback loop shown in Fig. y

$$- A(P)\beta(P) = 0 \qquad (5$$

eliminate the possibility of uncontrolled oscillations, it will be necess to investigate the frequency-response characteristics of typical d-c amp fiers in detail.

The Effects of Tube and Circuit Capacitance. Computation of Sta Transfer Functions. With d-c amplifiers, no oscillations due to excessiv phase shifts are experienced at low frequencies, but careful design i necessary in order to prevent regeneration at frequencies higher than th

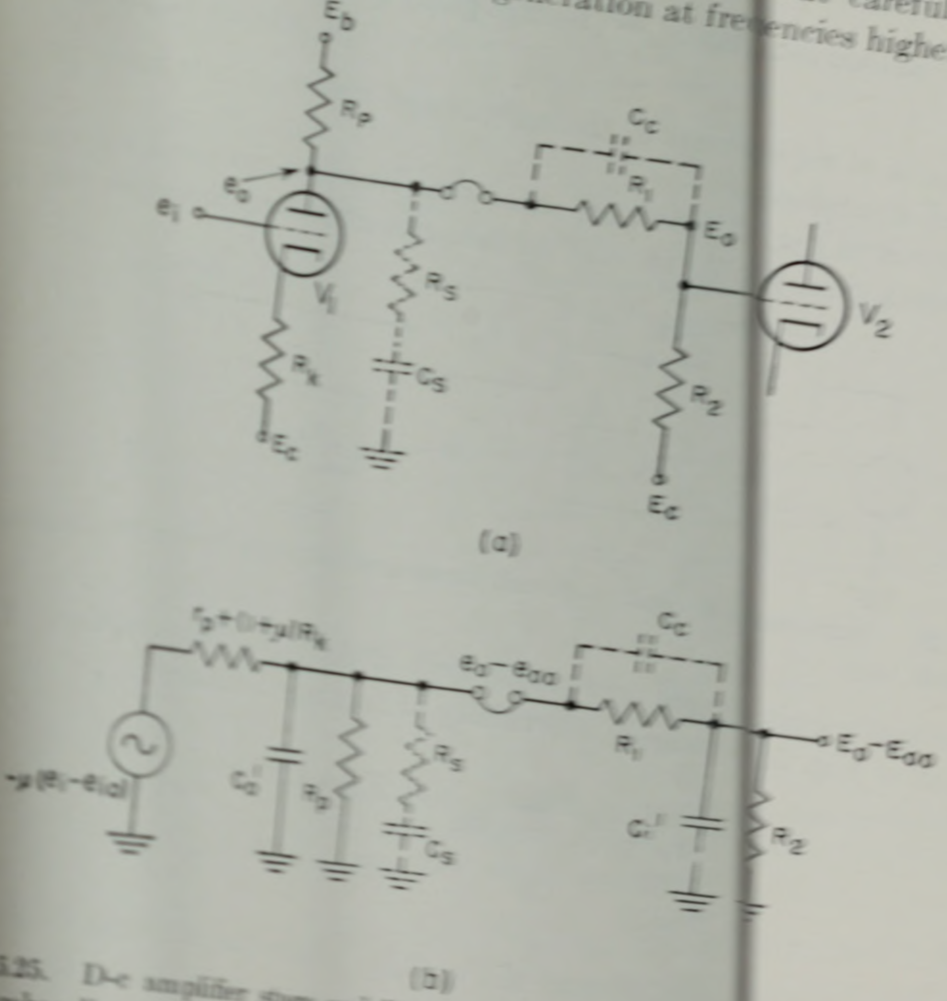

Fig. 5.25. D-c amplifier stage and linear equivalent circuit at high frequencies. The broken lines show high-frequency stabilization networks.

ctual operating frequencies. At such frequencies, th [illegible] rcuit capacitances must be taken into considerati [illegible]

A typical simple d-c amplifier stage like that sho [illegible] nd 25a will be analyzed by reference to its linear eq [illegible] e possible to analyze the high-frequency behavi [illegible] rcuits such as cathode followers or differenti [illegible] nalogous manner.

The high-frequency equivalent circui [illegible] 25a is shown in solid lines in Fig [illegible] familiar circuit of Fig. 5.7*b* with the addition of the capacitances C_o' and C_i'. C_o' is the output capacitance of the tube V_1 given in the tube manual plus an allowance of about 5 to 10 $\mu\mu$f for distributed circuit capacitance. C_i' is the input capacitance of the succeeding stage also plus an allowance of 5 to 10 $\mu\mu$f for circuit capacitances. If V_2 is a triode, a portion of the input capacitance will be due to the Miller effect.[1] In this case

$$C_i' = \text{grid-cathode capacitance} + (\text{1-stage gain})\ \text{grid-plate capacitance} + \text{circuit capacitance}$$

The effects of capacitances in the *cathode and screen circuits* have been considered as negligible.

The circuit of Fig. 5.25*b* will act somewhat like two cascaded integrating networks (Sec. 4.3); *the effect of the tube and circuit capacitances will be to introduce gain changes and phase shift at the higher frequencies.* The circuit of Fig. 5.25*b* consists essentially of four admittances[2]

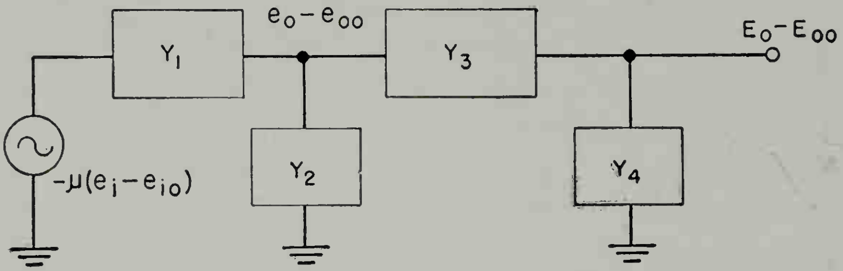

FIG. 5.26. More general equivalent circuit for an amplifier stage having no subsidiary feedback loop.

$$\left.\begin{aligned} Y_1 &= \frac{1}{r_p + (1+\mu)R_k} & Y_2 &= C_o'P + \frac{1}{R_p} \\ Y_3 &= \frac{1}{R_1} & Y_4 &= C_i'P + \frac{1}{R_2} \end{aligned}\right\} \tag{5.17}$$

arranged in the general circuit of Fig. 5.26. Since the currents flowing to any point of the circuit must add up to zero, one obtains the nodal equations

$$\begin{gathered} [(e_o - e_{o0}) + \mu(e_i - e_{i0})]Y_1 + (e_o - e_{o0})Y_2 + [(e_o - e_{o0}) - (E_o - E_{o0})]Y_3 = 0 \\ [(E_o - E_{o0}) - (e_o - e_{o0})]Y_3 + (E_o - E_{o0})Y_4 = 0 \end{gathered} \tag{5.18}$$

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[1] Miller effect is a common term for the amplification of the grid-plate capacitance in certain amplifier stages. The effect is analogous to the operation of the parallel-feedback integrator discussed in Sec. 4.5.

[2] The *admittance* Y of any two-terminal circuit is the ratio of corresponding current changes through the circuit to voltage changes across the circuit. The admittance is thus the reciprocal of the *impedance* of the circuit.

and hence the transfer function

$$\frac{E_o - E_{o0}}{e_i - e_{i0}} = -\mu \frac{Y_1Y_3}{Y_1Y_3 + Y_1Y_4 + Y_2Y_3 + Y_2Y_4 + Y_3Y_4} \tag{5.19}$$

Computation of the Amplifier Output Voltage and of the Feedback Transfer Function. In the computation of the amplifier output voltage and of the *feedback transfer function* $\beta(P)$, the effects of tube and circuit capacitances must also be taken into consideration. Figure 5.27*a* and *b* shows linear equivalent feedback circuits for a *summing amplifier* and a *summing integrator*, respectively; it is seen that, in the computation of the loop transfer function $A\beta$, the output stage and the feedback networks of a parallel feedback amplifier may be treated together just like the amplifier stages discussed above.

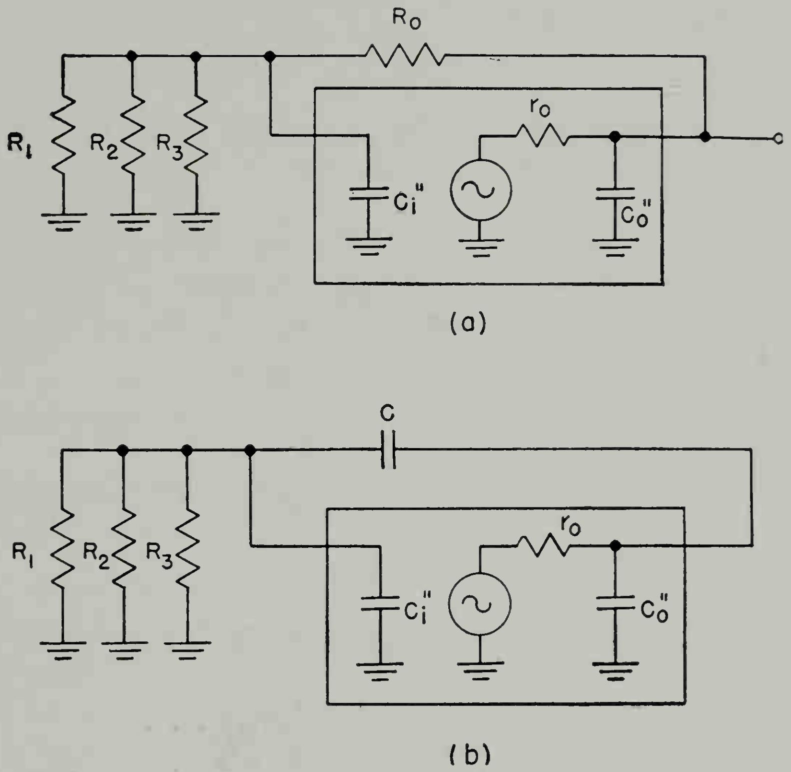

FIG. 5.27. Equivalent circuits for a summing amplifier (*a*) and a summing integrator (*b*), illustrating the effects of the effective output capacitance C_i'', the effective input capacitance C_i'', and the amplifier output impedance r_o.

Properties of the Loop Transfer Function. The loop transfer function $A\beta$ will be the product of the transfer functions (5.19) for each stage; the amplifier output stage and the feedback networks will be considered together as one stage. $A\beta$, as a function of P, will be of the general form

$$A\beta = \frac{f_1(P)}{f_2(P)} \tag{5.20}$$

where $f_1(P)$ and $f_2(P)$ are polynomials in P. It is possible to show that[1]

1. The degree of the polynomial $f_2(P)$ is always greater than the degree of the polynomial $f_1(P)$.
2. If the interstage networks contain only resistances and capacitances, and if no subsidiary feedback loops are used, then the roots s_i of
$$f_1(P) = 0$$
as well as the roots s'_k of
$$f_2(P) = 0$$
will be real and negative. The s_i are the *zeros* and the s'_k are the *poles* of the loop transfer function $A\beta$.

The first theorem simply expresses the fact that the loop gain must eventually decrease as the frequency of the input voltage is increased. The second theorem makes it possible to rewrite Eq. (5.20) in the form

$$A\beta = \frac{-k(P - s_1)(P - s_2) \cdots}{(P - s'_1)(P - s'_2) \cdots} \tag{5.21}$$

where k is a positive real constant. For steady sinusoidal voltages represented by

$$\tilde{X} = \bar{X}e^{j\omega\tau}$$

the complex loop gain $\bar{A}\bar{\beta}$ is obtained by replacing each P in Eq. (5.21) by $j\omega$:

$$\bar{A}\bar{\beta} = \frac{-k(j\omega - s_1)(j\omega - s_2) \cdots}{(j\omega - s'_1)(j\omega - s'_2) \cdots} \tag{5.22}$$

The absolute value of $\bar{A}\bar{\beta}$

$$|\bar{A}\bar{\beta}| = k\sqrt{\frac{(\omega^2 + s_1^2)(\omega^2 + s_2^2) \cdots}{(\omega^2 + s_1'^2)(\omega^2 + s_2'^2) \cdots}} \tag{5.23}$$

is the rms or peak loop gain for sinusoidal voltages of circular frequency ω. The phase shift Φ corresponding to Eq. (5.22) is

$$\Phi = \pi + \left[\text{arc tan}\left(\frac{-\omega}{s_1}\right) + \text{arc tan}\left(\frac{-\omega}{s_2}\right) + \cdots\right] - \left[\text{arc tan}\left(\frac{-\omega}{s'_1}\right) + \text{arc tan}\left(\frac{-\omega}{s'_2}\right) + \cdots\right] \tag{5.24}$$

The equations (5.23) and (5.24) permit one to plot the loop gain and the corresponding phase shift[1] against the circular frequency ω or against each other in order to determine whether the feedback amplifier is stable. This will be the case if Φ does not differ from 180 deg by more than 180 deg while $|\bar{A}\bar{\beta}|$ is greater than one.

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[1] Bode, *op. cit.*

High-frequency Stabilization Networks. Equation (5.24) shows that the phase shift associated with the loop transfer function (5.21) is intimately associated with the poles s'_k and zeros s_i of that function. Each pole s'_k contributes a phase lag Φ'_k of 45 deg at $\omega = -s'_k$. As ω increases further, this phase lag Φ'_k increases to 90 deg as given by

$$\Phi'_k = \text{arc tan} \left(\frac{\omega}{-s'_k} \right)$$

Whereas each of the terms $(j\omega - s'_k)$ in the denominator of Eq. (5.22) also acts to decrease the loop gain with increasing frequency, this effect may not be sufficient to keep the amplifier stable if the d-c gain (gain for $\omega = 0$) is high. Each zero s_i of the function (5.21), on the other hand, contributes a negative phase lag which can partially cancel the phase lag due to the poles s'_k; each zero s_i contributes a (negative) phase lag

$$\Phi_i = -\text{arc tan} \left(\frac{\omega}{-s_i} \right)$$

equal to -45 deg for $\omega = -s_i$ and tending to -90 deg with increasing ω.

The values of the zeros s_i and the poles s'_k are relatively easily computed from the transfer functions of each stage. An inspection of the expression (5.19) for a stage transfer function shows that each stage of the type shown in solid lines in Fig. 5.25*a* can contribute two poles s'_k; zeros can be contributed only by feedback networks such as those of the integrator shown in Fig. 5.27*b*.

A practical method for canceling some of the phase lag due to the poles s'_k of the loop transfer function is to introduce suitable zeros artificially. The dotted lines in the d-c amplifier circuit of Fig. 5.25*a* and *b* show two *high-frequency stabilization networks* (lead networks) used for this purpose. The by-pass capacitor C_c modifies the admittance Y_3 in Eqs. (5.17), which becomes

$$Y_3 = \frac{1}{R_1} + C_c P \qquad (5.25)$$

---

[1] The loop gain and the phase shift are not independent of each other; the relations existing between these two quantities have been made the basis of a very elegant graphical analysis method. See H. M. James, N. B. Nichols, and R. S. Phillips, *Theory of Servomechanisms*, MIT Radiation Laboratory Series, Vol. 25, McGraw-Hill, New York, 1947.

whereas the network made up of R_s and C_s changes the original value of the admittance Y_2 to

$$Y_2 = \frac{1}{R_p} + C'_oP + \frac{C_sP}{R_sC_sP + 1} \tag{5.26}$$

These two networks may be used independently.

On substituting Eq. (5.25) into Eq. (5.19) it will be seen that C_c introduces a zero into the stage transfer function and thus also into the loop transfer function. The R_sC_s network will introduce a zero but also a new pole; in general, the latter may, however, be made to correspond to so high a circular frequency ω that it will do no harm.

It will be desirable to modify the circuits so that the values of the zeros and the first poles of the loop transfer function $A\beta$ alternate as shown by the sequence

$$s'_1 < s_1 < s'_2 < s_2 < s'_3 \cdot \cdot \cdot$$

If this is true, and if there are less than three more poles than there are zeros, the phase shift will differ by less than 180 deg from 180 deg for all finite values of the circular frequency ω, and the amplifier will be stable for *all* negative values of the d-c gain. This goal cannot always be achieved in practice, but stabilization networks may be employed in the majority of cases to stabilize d-c amplifiers for all practical values of the d-c gain. A practical design method will be presented below.

Practical Design Method. The computation of the poles and zeros of each stage transfer function from the expression (5.19) is straightforward but somewhat tedious. The following is a simplified method for the design of high-frequency stabilization networks.

In single-stage amplifiers, no stabilization will be needed. In d-c amplifiers having two or more stages, *the first step is to eliminate the phase lag due to the interstage coupling networks* (R_1, R_2, and C'_i in Fig. 5.28*a*) by means of a by-pass capacitor C_c. The modified interstage coupling network, shown in Fig. 5.28*a*, has the transfer function

$$\frac{E_o - E_{o0}}{e_o - e_{o0}} = \frac{R_2}{R_1 + R_2} \frac{1 + R_1C_cP}{1 + \dfrac{R_1R_2}{R_1 + R_2}(C_c + C'_i)P} \tag{5.27}$$

If the value of C_c is chosen so that

$$C_c = C'_i \frac{R_2}{R_1} \tag{5.28}$$

Eq. (5.27) reduces to

$$\frac{E_o - E_{o0}}{e_o - e_{o0}} = \frac{R_2}{R_1 + R_2} \tag{5.29}$$

The only effect of C_c and C_i' is now to add to the capacitance C_o' loading the first tube. The new simplified equivalent circuit for the stage is shown in Fig. 5.28*b*. In practice, the value of the capacitance C_c should be chosen slightly in excess of that given by Eq. (5.28), so that all phase lag due to the interstage network will be eliminated with certainty.

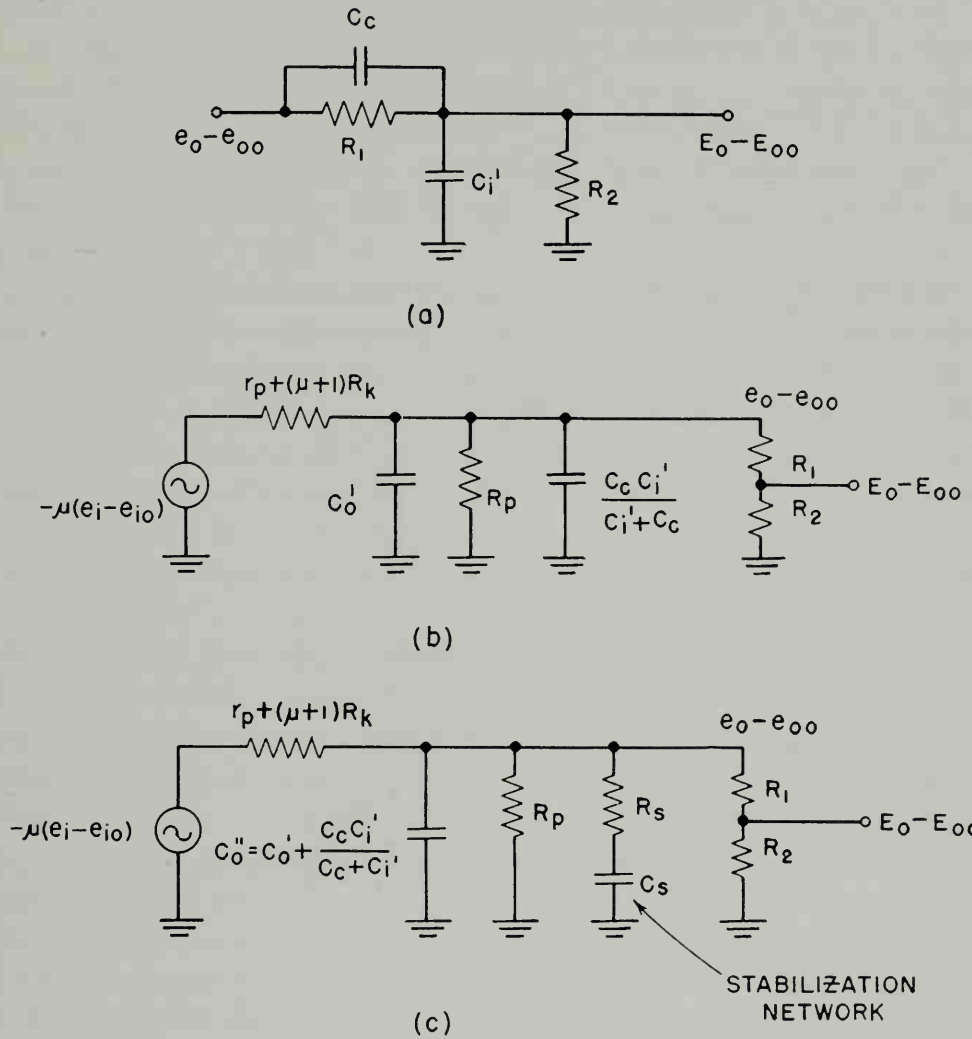

FIG. 5.28. Design of high-frequency stabilizing networks.

After this simplification it will be much easier to compute each stage gain. For a two-stage amplifier, no further stabilization will be necessary. For amplifiers having more than two stages, further stabilization may be needed. Figure 5.28*c* shows the simplified equivalent circuit of Fig. 5.28*b* with the addition of a stabilization network comprising the resistance R_s and the capacitance C_s. If Eq. (5.28) is assumed to hold, the stage transfer function is

$$\frac{E_o - E_{o0}}{e_i - e_{i0}} = \frac{-K\left(P + \frac{1}{C_sR_s}\right)}{P^2(r_p'R_sC_o''C_s) + P(R_sC_s + r_p'C_o'' + r_p'C_s) + 1} \tag{5.30}$$

where

$$C_o'' = C_o' + \frac{C_c C_i'}{C_i' + C_c}$$

$$r_p' = \frac{R_p[r_p + (1 + \mu)R_k](R_1 + R_2)}{R_p\,(R_1 + R_2) + [r_p + (1 + \mu)R_k](R_p + R_1 + R_2)}$$

$$K = \frac{\mu r_p' R_2 R_s C_s}{(R_1 + R_2)[r_p + (1 + \mu)R_k]}$$

This is of the form

$$\frac{E_o - E_{o0}}{e_o - e_{o0}} = \frac{-k(P - s_1)}{(P - s_1')(P - s_2')} \qquad \text{with} \qquad s_1 = -\frac{1}{R_s C_s} \tag{5.31}$$

It is seen that one zero and two poles will be associated with each stage incorporating stabilization networks of the type described if (5.28) holds exactly (in general, there will be two zeros and three poles).

If it is permissible to assume that

$$R_s \ll r_p' \qquad \text{and} \qquad C_o'' \ll C_s \tag{5.32}$$

the poles s_1' and s_2' are given by

$$s_1' = -\frac{1}{R_s C_o''} \qquad s_2' = -\frac{1}{r_p' C_s} \tag{5.33}$$

With the location of the poles and zeros given by such relatively simple formulas, it is not too difficult to choose the correct values of the circuit constants for each stage.

Numerical examples showing the design of high-frequency stabilizing networks are found in several textbooks on amplifier design.[1] As an interesting exercise, the reader might compute the zeros and poles of the loop transfer function for a summing amplifier using the circuit of Fig. 5.33.

All stabilization networks for feedback amplifiers must be designed with large safety factors (perhaps 10-*db amplitude margin) in order to allow for tolerances in the amplifier components as well as for the usually rather rough estimates of the distributed capacitances.* Each circuit must be tested for stability; if the initial design was sound, any remaining oscillations can usually be eliminated experimentally by shunting one or more of the interstage coupling resistors (R_1 and R_2 in Fig. 5.25*a*) with small capacitors.

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[1] Greenwood, I. A., J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948; James, Nichols, and Phillips, *op. cit.*

Whenever stabilizing networks are introduced in operational amplifiers, care should be taken that they do not alter the transfer function of the operational amplifier within the range of expected signal frequencies.

Stability Analysis of More Complicated Circuits. In principle, all linear feedback circuits may be analyzed by means of *equivalent circuits* in the manner outlined above. A number of elegant graphical methods have been developed to facilitate such analyses; the reader is referred to the textbooks[1] on this subject. A detailed discussion, *e.g.*, of the stability of *regenerative amplifiers*, is beyond the scope of this book.[2] A tendency to high-frequency instability in regenerative d-c amplifiers may often be combated by decreasing the amount of regeneration at the higher frequencies.

Over-all Stability of Computer Setups. The stability of individual computing elements is no guarantee for the high-frequency stability of computer setups consisting of cascaded computing elements. Quite frequently, loops of computing elements with loop gains in excess of 20 will break out into uncontrolled high-frequency oscillations. This is due to the addition of small phase shifts associated with each computing element and possibly also to stray coupling between computing elements. These considerations establish limitations of the computer setups possible, especially when fast time scales are used.

5.5. Practical Examples of Computer Amplifier Design and Construction. *Introduction.* The following examples of practical d-c amplifier circuits used in computer applications will serve to illustrate the design principles presented in the preceding sections. A brief discussion of the physical construction of computer amplifiers is also given at the end of this section. Examples of computer amplifiers using automatic balancing are shown in Sec. 5.6.

One-tube Amplifiers and Integrators. Figure 5.29 shows a *simple pentode amplifier* suitable for use in applications which do not require great accuracy, *e.g.*, in simple repetitive computers. The gain of this circuit is about 150. No special circuits for drift reduction are employed.

The *regenerative integrator circuit* shown in Fig. 5.30 was developed at the Massachusetts Institute of Technology[3] and is capable of good integration. The circuit may be quite susceptible to drift and distortion; in the version shown, moreover, the output voltage reference level is not the same as the input reference level. But with a suitable clamping device

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[1] *Ibid.*

[2] In amplifiers containing subsidiary feedback loops, the poles and zeros of the loop transfer function need no longer be real but may occur in complex conjugate pairs. Each pair can give rise to phase shifts in excess of 90 deg.

[3] Greenwood, Holdam, and MacRae, *op. cit.*

(see Sec. 8.5), this circuit might make a satisfactory integrator for a low-cost repetitive computer; in this application, only one power supply is needed.

Multistage Amplifier Circuits Requiring Only Two Power Supplies. The simple two-tube d-c amplifier circuit shown in Fig. 5.31, developed at the University of Michigan,[1] incorporates a differential amplifier input stage for drift reduction, a triode voltage amplifier stage, and a cathode-follower output stage. Only one tube type is used, and only two power supplies are needed. The gain of this amplifier is about 700, and the output impedance is 700 ohms without the over-all feedback circuits. The amplifier will deliver plus or minus 40 volts to a load of not less than 25,000 ohms resistance. The long-time drift is of the order of 1 mv in 5 hr.

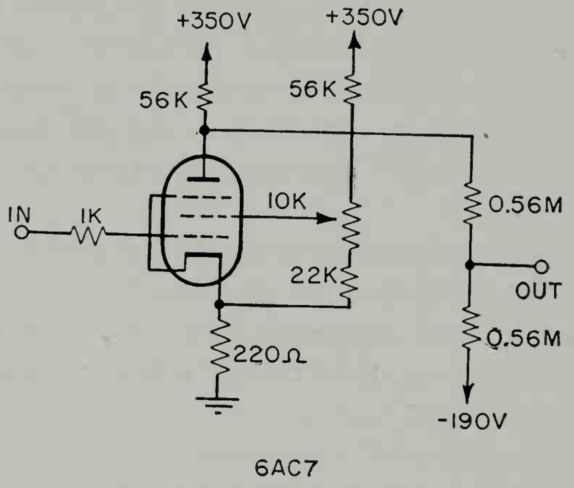

FIG. 5.29. Simple one-tube d-c amplifier.

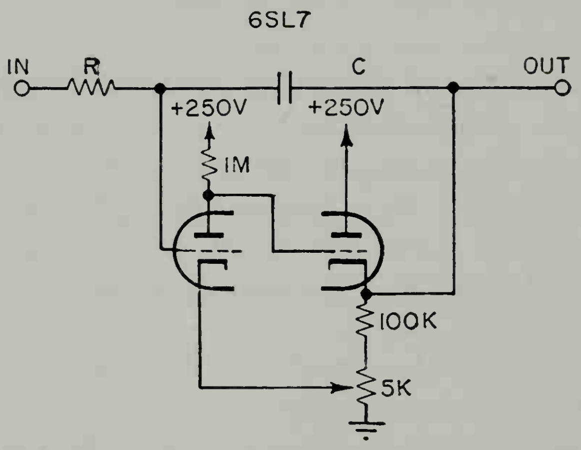

FIG. 5.30. Simple regenerative integrator with cathode-follower output. (Output is not at the same reference level as input.)

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[1] McDonald, D., Analog Computer for Servo Problems, *Rev. Sci. Instruments*, **21**: 154, 1950.

The new low-cost analog computer of the Boeing Aircraft Company uses the d-c amplifier circuit shown in Fig. 5.32. This is a particularly interesting design comprising two differential amplifier stages which permit drift reduction, cathode bias without any loss in gain, and regenerative feedback around the second stage. Cathode-follower output employing only one section of the output 6SN7 is used ordinarily; by using both sections of the 6SN7 in parallel, it is possible to drive as many as twelve 50,000-ohm potentiometers through the output voltage range of plus or minus 50 volts. The amplifier is said to be capable of driving a Brush recording oscillograph directly (without a recorder driver) with fair accuracy. The designers have made provisions for connecting an optional automatic balancing unit (see Sec. 5.6) to the amplifier. Especially with the addition of such a device, this d-c amplifier should be capable of excellent performance in computer applications. Note that the interstage by-pass capacitors (see Sec. 5.4) are the only stabilization networks needed in this circuit. The frequency response of this all-triode amplifier is said to make it suitable for use in repetitive computers (at a repetition rate just sufficient for eye persistence) as well as in "slow" d-c analog computers.

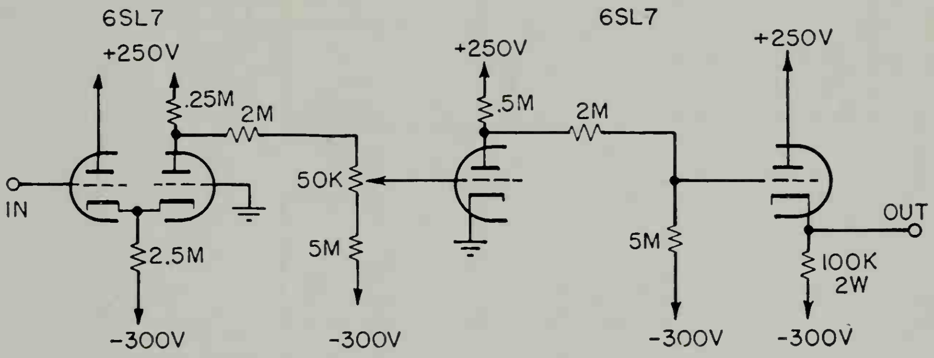

FIG. 5.31. University of Michigan d-c amplifier.

Examples of High-gain D-c Amplifiers Using Three Power Supplies. In the amplifier circuit of Fig. 5.33, high voltage gain is achieved without the use of regeneration through the use of a pentode in the second stage and a beam power tube output stage which also has some voltage gain. The output voltage range is plus or minus 100 volts with a load of about 15,000 ohms. A differential amplifier stage arranged as a Miller compensator (see Sec. 5.3) is used as the input stage and serves to minimize the effects of cathode emission changes. The plate voltages for the input stage are kept low in order to minimize grid current.

Three high-voltage power supplies are required for the operation of this amplifier; the plus 75 volts indicated in the diagram are obtained from the plus 350-volt supply by means of a voltage divider using a VR75

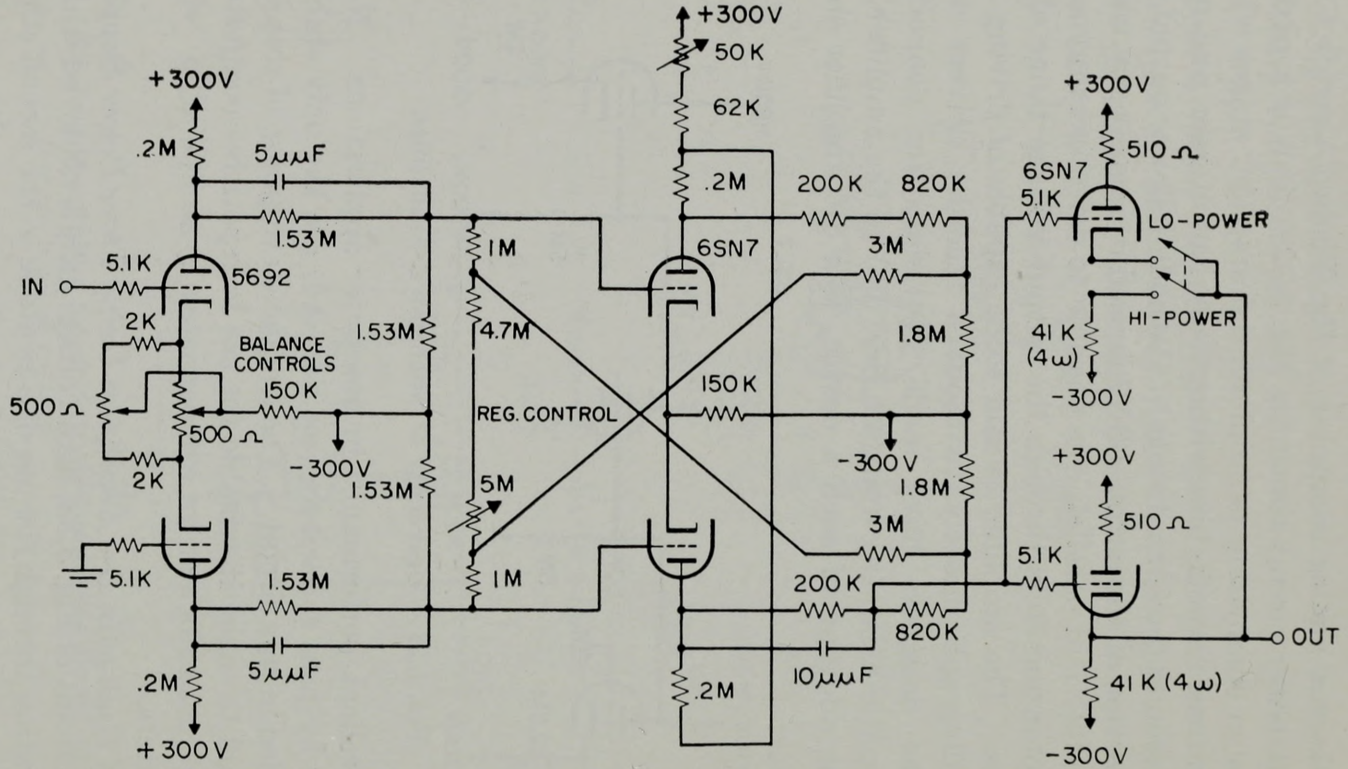

FIG. 5.32. Boeing Aircraft Company d-c amplifier circuit. The output switch connects either one or both of the 6SN7 triode sections with the output circuit, depending on the power required.

voltage-regulator tube. This amplifier was designed by the Bell Telephone Laboratories for the M-9 fire-control computer; this design has served as a model for many other computer amplifier circuits. The gain obtainable with such arrangements varies from 10,000 to 75,000 without the use of regeneration.

The two Goodyear Aircraft Corporation computer amplifier circuits (Figs. 5.34 and 5.35) are both fine examples of elegant d-c amplifier design. In the circuit of Fig. 5.34, three cascaded differential amplifier stages permit high voltage gain with low drift. The second stage is arranged to compensate for the effects of voltage changes in the plate supply. A pentode output stage is used and permits an output voltage range of plus or minus 100 volts with a 20,000-ohm load.

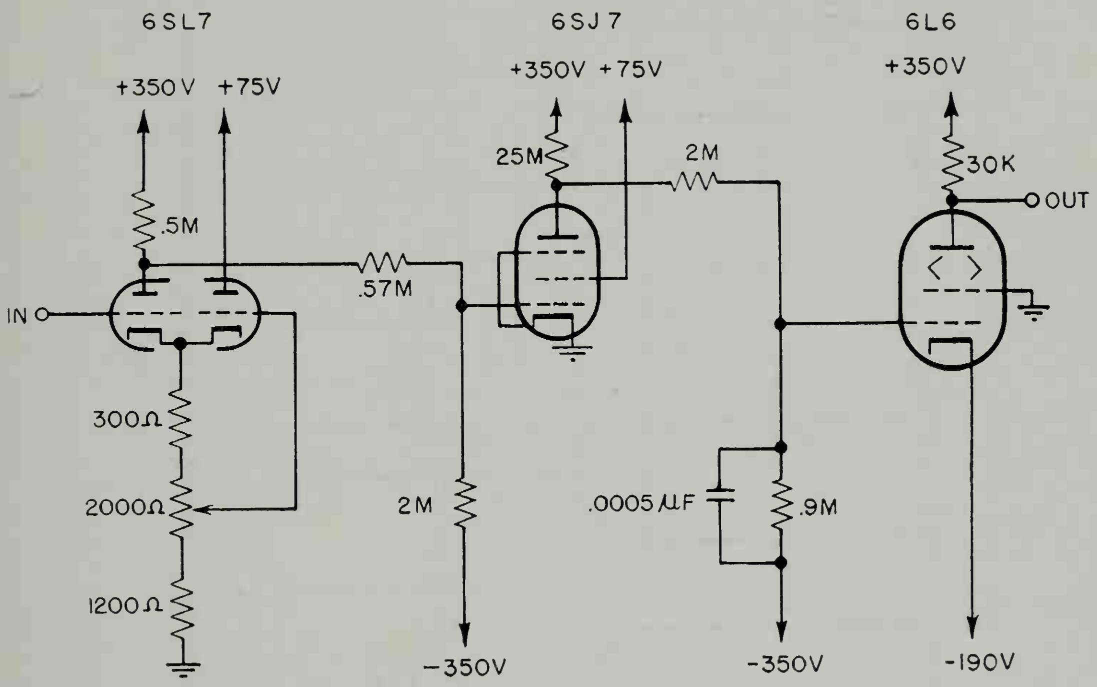

FIG. 5.33. Computer amplifier designed by the Bell Telephone Laboratories for the M-9 fire-control computer.

The second Goodyear amplifier, shown in Fig. 5.35, has a series-compensated input stage. Two cascaded differential amplifier stages are used to amplify the difference between the output voltage of the series-compensated stage and the corresponding reference level. The output tube is, again, a pentode. The input stage is operated at reduced filament voltage for improved stability and tube life[1] without any sacrifice in gain. The grid current in the input stage is minimized through reduced

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[1] Maintenance of the Electronic Analog Computer, Goodyear Aircraft Corporation, *Rept.* GER 3046, July 15, 1949.

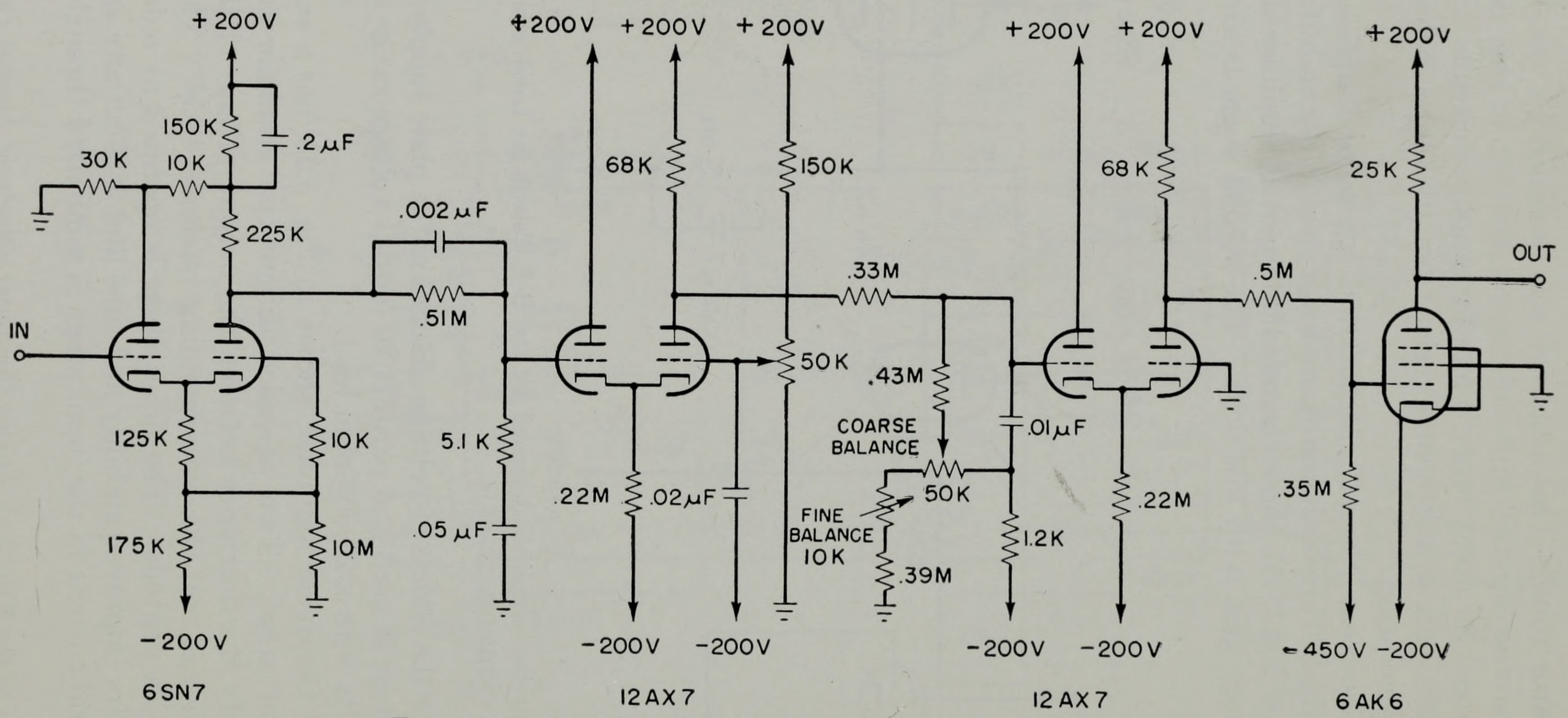

Fig. 5.34. Goodyear Aircraft Corporation computer amplifier.

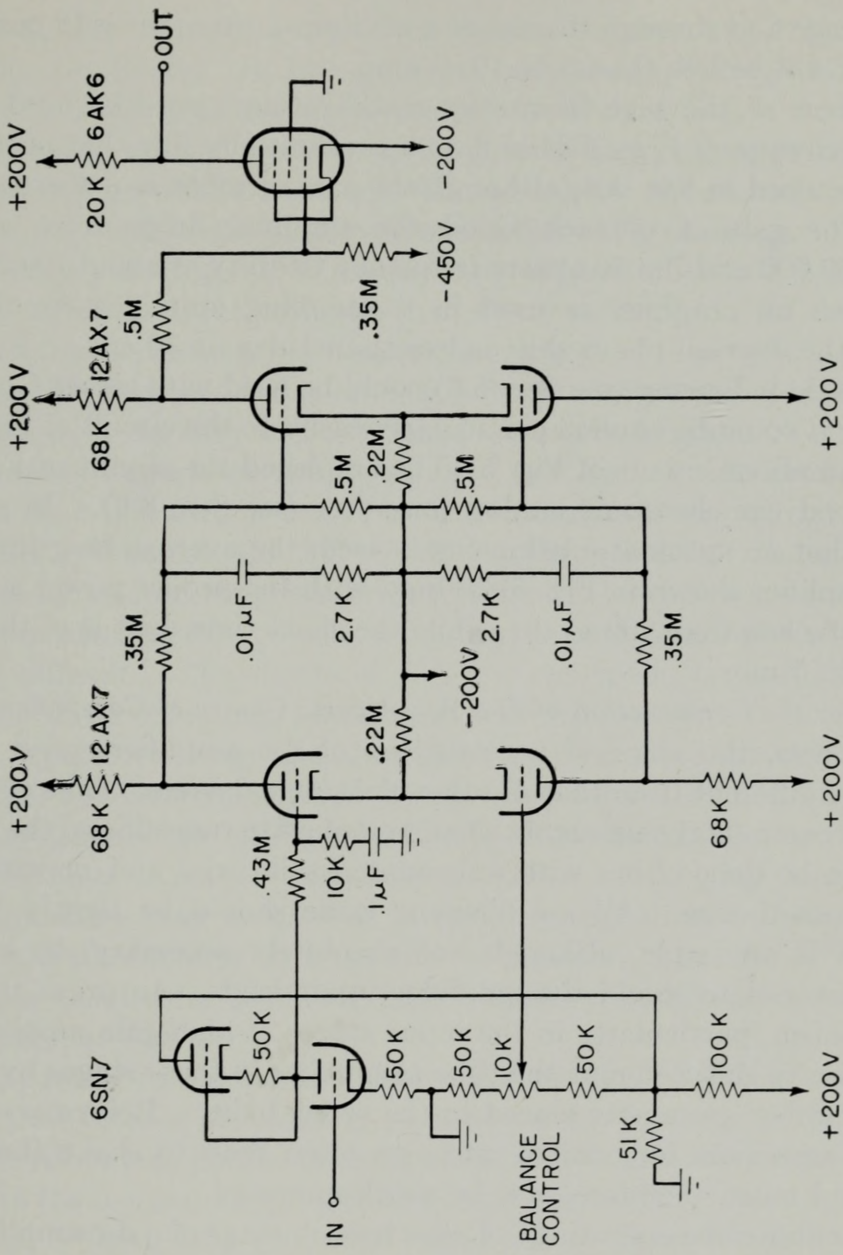

FIG. 5.35. Goodyear Aircraft Corporation computer amplifier.

plate voltage and through the use of a medium-μ tube; the grid current in this stage will be less than 5×10^{-11} amp.

The effect of the high-frequency stabilization networks used in the amplifier circuits of Figs. 5.34 and 5.35 is essentially like that of the networks discussed in Sec. 5.4, although the arrangement is not exactly the same. The gain A of each Goodyear amplifier drops from a value between 50,000 and 70,000 at zero frequency to unity at about 100,000 cps. When such an amplifier is used in a summing amplifier circuit with $\beta = 0.9$, the over-all phase shift is less than 1 deg at 30 cps.

Automatic balancing (see Sec. 5.6) could be used with either Goodyear amplifier; it could be applied particularly easily to the circuit of Fig. 5.34. The d-c amplifier circuit of Fig. 5.35 has replaced the circuit of Fig. 5.34 in the Goodyear electronic analog computer (see Sec. 8.6). In spite of the fact that no automatic balancing is used, the average long-time drift of the amplifier shown in Fig. 5.35, used with the proper power supplies, is said to be less than 200 μv/hr, while the short-time drift is of the order of 50 μv in 5 min.

Hints for the Construction of D-c Amplifiers: Choice of Components. In most respects, the physical construction of d-c amplifiers need not be essentially different from that of other electronic devices. The construction of all computer components should emphasize ruggedness; the chassis wiring can be done either with uninsulated solid wire and/or with insulated stranded wire. All a-c filament leads should be tightly twisted pairs. It is advisable, although not absolutely necessary, to keep all leads short and to shield the amplifier input leads. In order to avoid microphonism, particularly in the input stages of high-gain amplifiers, it will be wise to shock-mount the tube assemblies in these stages by means of small rubber grommets placed in the screw holes. Relays associated with d-c amplifiers for control purposes often tend to shake the entire chassis and must, therefore, also be shock-mounted.

The vacuum-tube operating voltages in each stage of a d-c amplifier are affected by resistors in preceding stages as well as in the stage in question. For this reason, the resistances of all resistors carrying direct current should be specified to within plus or minus 1 per cent.

In order to minimize d-c amplifier drift, all resistors carrying direct current must be *stable: i.e.*, they must maintain their resistance values within specified tolerances over reasonable ranges of current, humidity, and ambient temperature. In order to avoid excessive temperature changes, all resistors used in computer amplifiers should have about twice the power rating actually required; resistors which must dissipate more than 1 or 2 watts may be mounted on top of the amplifier chassis for better heat dissipation.

High-quality metalized-type resistors are recommended for low-cost computer amplifiers. *In high-precision computers, all critical resistors (in the power-supply regulators and servomechanisms as well as in the d-c amplifiers) should be special precision types of metalized, carbon, or wire-wound resistors.*[1] Wire-wound resistors will not exhibit the required temperature stability unless special alloys are used for the windings.

Generally speaking, it is usually less expensive to use high-quality resistors in all doubtful positions than to try to compute the drift due to the resistance variations of cheaper resistors.

Adjustable resistors and balancing potentiometers must also be chosen with a view to stability and freedom from microphonism.

Integrating Capacitors: Absorption and Leakage. It has been shown in Chap. 4 that *high leakage resistance* is a desirable property in integrating capacitors. Whereas the effects of capacitor leakage could be compensated by means of regenerative circuits, the leakage resistance of poor capacitors tends to change with humidity, so that accurate compensation may be difficult. The effects of *dielectric absorption* (soak-in) are somewhat like those of leakage but change with frequency.

Both capacitor leakage and absorption are minimized through the use of the proper types of capacitors. Integrating capacitors in high-accuracy computers should use either polystyrene or similar plastics as the dielectric[2] and have leakage resistances in excess of 5×10^5 megohms/μf. Mica capacitors may be used as integrating capacitors in low-cost computers and particularly in repetitive computers; they are, incidentally, also recommended for use in the high-frequency stabilization networks of d-c amplifiers.

In all parallel-feedback-type integrators, care should be taken to confine the leakage path between the plates of each integrating capacitor to the dielectric proper. All circuit points, such as jacks, switch and relay contacts, connected to the d-c amplifier input and output terminals, respectively, must *always* be separated by grounded metal parts. Such construction practice will eliminate leakage paths over the surface of insulating materials. The analysis of Sec. 4.5 shows that leakage to ground is essentially harmless; "insulation with metal" is, therefore, the best possible insulation.

Amplifier Loading and the Use of Booster Resistors. For a given d-c amplifier, the range of output voltages not to be exceeded for proper linear operation will be specified for a given minimum load resistance. If the use of smaller load resistances is necessary (*e.g.*, if several potentiometers in parallel are to be connected to an amplifier output), the need for additional operational amplifiers in parallel with the first can sometimes be avoided through the use of *booster resistors*. A booster resistor is simply a resistor connected in parallel with the output plate load resistor of the amplifier output stage in circuits like those shown in Figs. 5.33 to 5.35. Such a resistor can be introduced into the circuit by means of a simple switch or plug-in arrangement. A booster resistor will decrease the amplifier output impedance at a slight expense in terms of amplifier forward gain and d-c power drain.

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[1] "Noblelloy" resistors (Continental Carbon Company), Shallcross types BX181E and BX190E, Carbofilm type CP-1, and IRE type DCH have been used successfully in d-c analog computers, but other types may also be useful.

[2] Western Electric type polystyrene capacitors and Condenser Products Company types LA (polyethylene) and PA (polystyrene) have been found useful.

5.6. Improved D-c Amplifier Design: Automatic Balancing Circuits. *Introduction.* The development of improved d-c amplifiers and d-c analog computers has received a tremendous impetus through the invention of *automatic balancing circuits*[1] which make it possible to obtain

1. A reduction of d-c amplifier drift (see Sec. 5.1) by a factor of 3 to 20.
2. An increase of the d-c amplifier gain at low frequencies by a factor of 1,000 to 5,000, resulting in decreased distortion and improved accuracy of integration.
3. The elimination of the necessity for frequent manual balancing. Balancing controls may be eliminated entirely at the price of a very slight reduction in accuracy.

Such possibilities permit striking improvements in the design of d-c analog computers. This section will present a discussion of the principles of automatic balancing circuits as well as a number of practical examples.

Modulated-carrier-type Circuits for D-c Amplification. Amplifier circuits of the modulated carrier type are frequently used instead of conventional d-c amplifiers in connection with various instrumentation applications.

Figure 5.36*a* shows the block diagram of such a circuit. The d-c input is used to modulate an a-c "carrier" voltage, so that the a-c amplitude is proportional to the d-c input voltage. The modulated carrier is amplified by means of a conventional a-c amplifier. The amplified alternating current is then demodulated to yield an amplified d-c output voltage proportional to the original d-c input voltage.

Figure 5.36*b* illustrates the use of a single *synchronous vibrator or chopper* for both modulation and demodulation. The d-c input voltage is interrupted periodically by the chopper relay, so that a modulated a-c voltage is obtained at the amplifier input. The amplifier output voltage is rectified synchronously by the same vibrator; the d-c output voltage is then smoothed by a smoothing network. It should be noted that such a modulated-carrier-type amplifier is *phase-sensitive;* if the sign of the input d-c voltage is reversed, the phase of the modulated a-c voltage changes by 180 deg and the sign of the d-c output voltage will also reverse in the correct manner.

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[1] Williams, A. J., R. E. Tarpley, and W. R. Clark, *Trans. AIEE*, **67**: 47, 1948; Goldberg, E. A., Stabilization of D-c Amplifiers, *RCA Rev.*, **11**: 296, 1950. To the best of the authors' knowledge, automatic balancing circuits were first used in the RCA "TYPHOON" computer.

Modulated-carrier-type amplifiers permit d-c amplification with very substantial gain, without the use of carefully regulated power supplies.

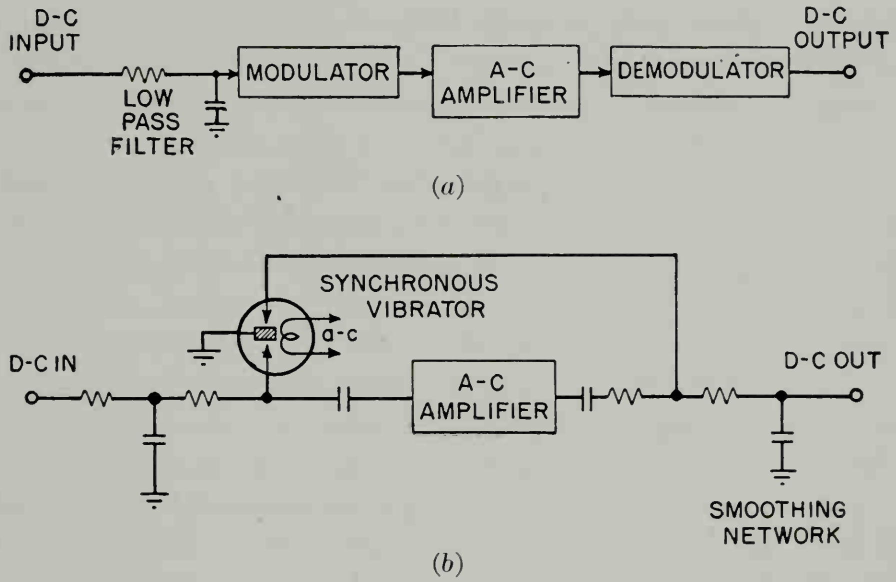

FIG. 5.36. Principle of the modulated-carrier-type d-c amplifier. In the circuit of (*b*), a single synchronous vibrator serves as both a phase-sensitive modulator and demodulator.

The use of synchronous vibrators in the manner shown in Fig. 5.36*b* permits one, moreover, to obtain d-c amplification with very low drift.

A synchronous vibrator is essentially an a-c-driven relay which can perform switching operations at carrier frequencies between 0 and 600 cps. Aside from possible contact and thermoelectric potentials, the main source of error voltages in synchronous vibrators is stray coupling of the driving a-c voltage into the signal circuit. This effect must be minimized by careful shielding and also by appropriate phasing of the driving voltage; it can also be reduced by grounding the adjustable center tap of a potentiometer connected across the relay coil. It may be useful to bring the coil terminals out through the top of the converter case, well away from the signal-carrying leads. Finally, it is possible to make the contact reed of a synchronous vibrator vibrate at twice the excitation frequency applied to the relay coil; such an arrangement will further reduce the effects of a-c pickup at the excitation frequency.

The value of possible noise and offset voltage output of a synchronous vibrator depends on the impedance level at which the device is operated and may be less than 2 μv for a synchronous vibrator operated into the low-impedance primary of an input transformer and 2 to 5 μv for a synchronous vibrator operated into a vacuum-tube grid at an impedance level of the order of 1 megohm. Good results should also be possible if the synchronous vibrator is replaced by a cam-operated or rotating switch actuated periodically through a high-speed motor. Offset and noise voltages lower than 0.005 μv have been obtained with an input resistance of 5 ohms.[1] Conceivably, a single motor could be made to drive all the modulating switches in installations comprising many such devices.

The range of signal frequencies usable with modulated-carrier-type amplifiers is restricted to frequencies lower than the carrier frequency by about a factor of 10. If modulated-carrier-type amplifiers are to be employed in feedback circuits, the high-frequency components must be eliminated through the use of low-pass filters (indicated in Fig. 5.36). Otherwise, the high-frequency components of the signal and noise would not modulate the carrier properly but might be amplified in the a-c amplifier and give rise to spurious high-frequency feedback. Since synchronous vibrators cannot be operated consistently at carrier frequencies much higher than 400 cps, it has not been possible to employ modulated-carrier-type amplifiers using synchronous vibrators directly in high-gain operational amplifiers for computer applications. Higher carrier frequencies can be utilized by replacing the synchronous vibrators with diode-type switching circuits.[2] Since the drift in the best diode-switch-type demodulators is not very much better than that observed with high-quality d-c amplifiers, no one has hitherto considered it worth while to use modulated-carrier-type amplifiers in d-c analog computers. This may be due to circuit complications and shielding difficulties involved in using high carrier frequencies.

It is conceivably possible to utilize a modulated-carrier-type amplifier channel employing a synchronous vibrator to amplify the d-c and low-frequency components of the signal and to amplify the higher frequency components in a parallel a-c amplifier channel. Such developments have been rendered obsolete by the invention of the automatic balancing circuits described below.

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[1] Liston *et al.*, *Rev. Sci. Instruments* **17**: 194, 1946.

[2] See B. Chance, F. C. Williams, V. Hughes, D. Sayre, and E. F. MacNichol, *Waveforms*, Chap. 12, MIT Radiation Laboratory Series, Vol. 19, McGraw-Hill, New York, 1949, for a discussion of switch type modulators.

Automatic Balancing Circuits. Figure 5.37 shows the block diagram of the basic automatic balancing circuit. A modulated-carrier-type amplifier using a synchronous vibrator is ingeniously combined with a conventional d-c amplifier in such a manner that the freedom from drift of the former and the superior high-frequency response of the latter are both realized. In addition, the circuit has a vastly increased forward gain at low frequencies; this feature is particularly useful for accurate integration (see Sec. 4.5).

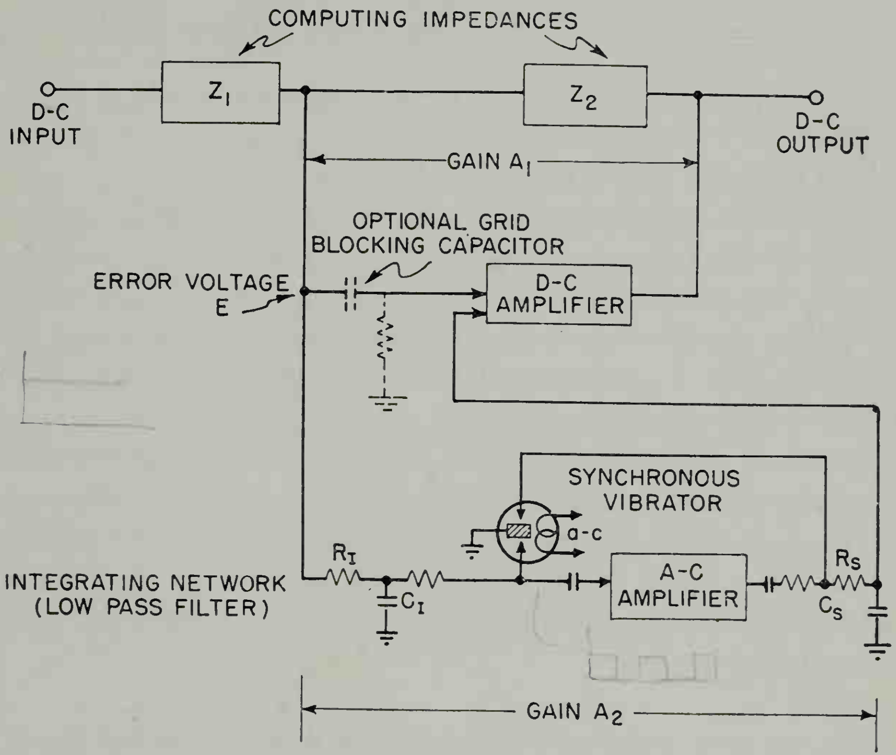

FIG. 5.37. Block diagram of an operational amplifier with automatic balancing circuit. The grid-blocking capacitor indicated by dash lines is an optional feature.

The time constant R_IC_I of the low-pass filter or integrating network comprising the resistance R_I and the capacitance C_I in Fig. 5.37 is usually chosen so that

$f_c \geq \frac{1.5}{R_IC_I}$

$$R_IC_I \geq \frac{1.5}{f_c} \tag{5.34}$$

where f_c is the carrier frequency to be used in the a-c amplifier (f_c will be between 60 and 500 cps).

As a result, the high-frequency components of the error voltage E are not affected by the automatic balancing circuit but are amplified by the d-c amplifier as in any operational amplifier circuit. The high-frequency forward gain of the operational amplifier shown is thus simply

$$A_{HF} = A_1 \tag{5.35}$$

where A_1 is the gain of the d-c amplifier channel.

If the circuit shown in the solid lines in Fig. 5.37 is used, the low-frequency components of the error voltage E are also amplified by the d-c amplifier. Much more significantly, however, components of the error voltage E of sufficiently low frequencies to pass easily through the low-pass filter will be amplified by the modulated carrier channel and the d-c amplifier in *cascade.* The low-frequency forward gain is thus seen to be

$$A_{LF} = A_1 + A_1A_2 \tag{5.36}$$

where A_2 is the gain of the modulated carrier channel.

The modulated carrier channel precedes the d-c amplifier as far as the d-c and low-frequency components of the signal are concerned. Since this channel is not affected by voltage changes in the high-voltage and filament power supplies, it acts essentially as an input stage having a relatively high gain A_2 and very low drift. Reference to Eq. (5.2) shows that *such an arrangement reduces the effects of drift in the d-c amplifier channel effectively by a factor equal to the gain A_2 of the modulated carrier channel.* Although it will still pay to keep the drift in the d-c amplifier channel to a minimum by the techniques discussed in the previous section, *the most significant sources of drift will now be only those acting at the input to the modulated carrier amplifier channel.* These sources of drift are essentially twofold, *viz.,*

1. Noise and drift due to the synchronous vibrator, of the order of 2 to 20 μv.
2. Grid current of the input tube of the d-c amplifier; *e.g.,* a typical grid current of 10^{-4} μa through a resistance of 1 megohm will give rise to an error voltage of 100 μv. A portion of this error voltage will appear at the input of the modulated carrier channel and will act just like an input voltage.

Integral Control. The action of the automatic balancing circuit may be described in yet another manner. Any offset voltage due to drift in the d-c amplifiers, say, appearing at the output terminals of the operational amplifier will be fed back through the feedback networks and will tend to charge the capacitor C_I of the low-pass filter network at the input of the modulated carrier channel. This "low-frequency error voltage" will be amplified by the modulated carrier channel and fed back in such a manner as to anul the effects of the original offset voltage. It may be said, as a matter of fact, that the low-pass filter comprising the resistor R_I and the capacitor C_I "remembers" the effects of past disturbances for a short time and seeks to anul their effects. This type of action is referred to as integral control.[1]

Two Practical Circuits. Figure 5.38 shows the circuit diagrams of two practical computer amplifiers incorporating automatic balancing circuits. Both circuits were developed by the RCA Research Laboratories[2] under a contract with the United States Navy. The circuit of Fig. 5.38*a* employs a 60-cps carrier in the modulated carrier channel, whereas a 400-cps carrier is used in the circuit of Fig. 5.38*b*. The remarkable properties of amplifiers employing automatic balancing circuits are best illustrated by the following specifications of the circuit of Fig. 5.38*b*:

Gain A_1 of the d-c amplifier channel, about 50,000
Gain A_2 of the modulated carrier channel, about 2,000
Net gain at zero frequency, about 100 million (!)
Net gain at 5 cps, about 300,000
Grid current, less than 10^{-4} μa
Average offset voltage, referred to input, consistently less than 50 μv

The offset voltage can be further reduced by careful adjustment of the auxiliary balance control indicated in Fig. 5.38*b*, through the use of an electrometer-type tube in the input stage of the d-c amplifier or through the use of a grid-blocking capacitor, as described in the following paragraph.

Use of a Grid-blocking Capacitor.[3] *It is possible to eliminate the d-c offset voltage due to the constant component of the grid current in the d-c amplifier input tube through the use of a grid-blocking capacitor and separate grid return resistor.* Such an arrangement is shown in dotted lines in the block diagram of Fig. 5.37. With circuits of this type, the d-c component of the input signal cannot pass directly through the d-c amplifier channel but must pass first through the modulated carrier channel; the d-c gain will be equal to A_1A_2. Higher frequency components of the input signal can still pass directly through the d-c amplifier channel by way of the grid-blocking capacitor.

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[1] Refer to James, Nichols, and Phillips, *op. cit.*, for a detailed discussion of integral control.

[2] Princeton, N.J.

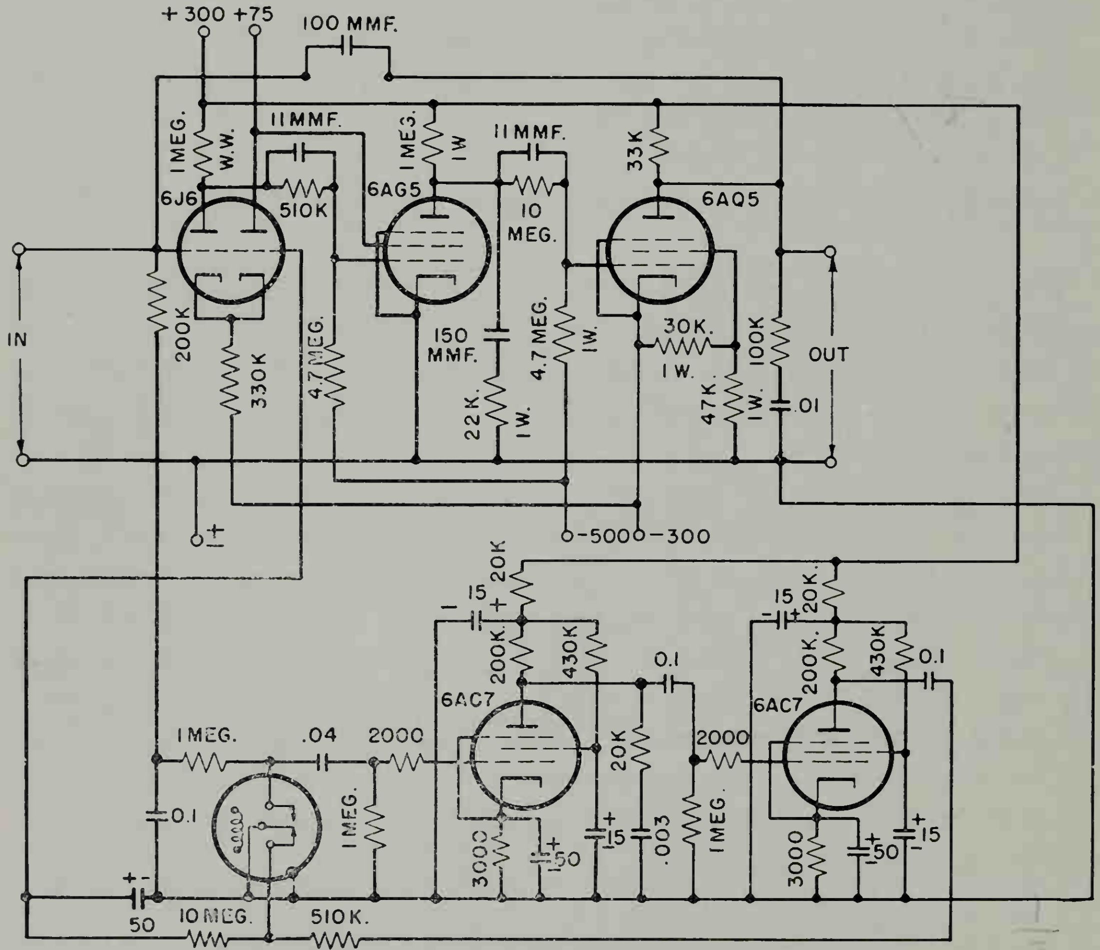
[3] Gunning, W. F. and A. S. Mengel, unpublished communication, RAND Corporation, Santa Monica, Calif.

The grid-blocking capacitor should be a high-quality-type capacitor having a minimum of leakage or absorption, such as a polyethylene or polystyrene capacitor.

The use of a grid-blocking capacitor permits a further reduction of d-c amplifier drift and is recommended for practical computer applications if the precautions outlined below are observed.

Recovery of Automatic Balancing Circuits after Overloads. Whenever the absolute value of the output voltage of an operational amplifier exceeds a certain permissible level, the amplifier will "saturate" and can then no longer function as a linear device. In this condition, the feedback cannot act to reduce the "error voltage" at the amplifier input terminals.

With conventional d-c amplifier circuits, normal operating conditions can be reestablished quickly after an overload by simply returning the



(*a*)

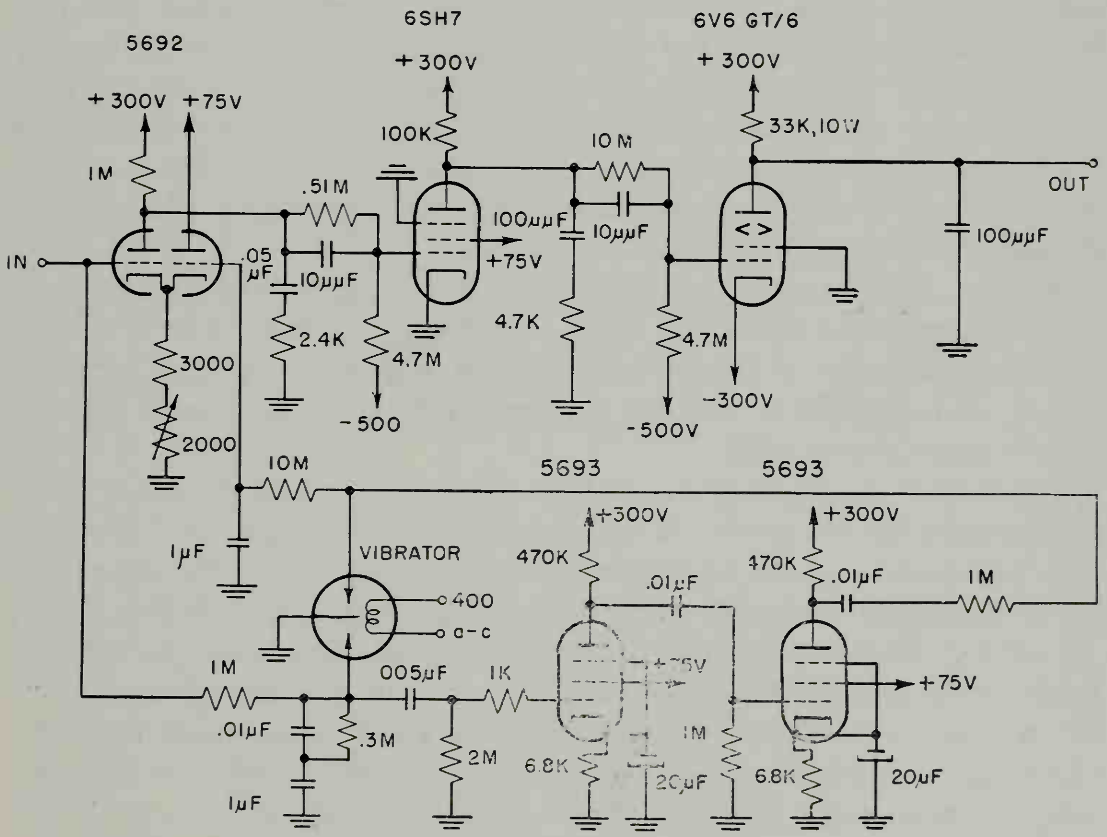
FIG. 5.38. Two high-quality d-c amplifiers incorporating automatic balancing circuits. Both circuits were developed at the RCA Research Laboratories, Princeton, N.J. The circuit of Fig. 5.38*a* uses a 60-cps carrier, and the circuit of Fig. 5.38*b* uses

computer to the RESET condition (see Sec. 7.1). In amplifiers employing automatic balancing circuits, however, the filter and grid-blocking capacitors used in these circuits will tend to charge up while the amplifier is overloaded. When the computer is returned to the RESET condition following an overload, some time may pass until these capacitors are discharged and permit the amplifier output voltage to return to its normal operating range. Consequently, provision should be made for discharging the capacitors in question after an overload.

A device for this purpose might be in the form of a relay which shorts the guilty capacitors for a short time whenever the computer is placed in the RESET condition (see Sec. 7.1).

The recovery time will, in any case, be shorter with automatic balancing circuits employing a 400-cps carrier than with 60-cps carrier systems, since the integrating and grid-blocking capacitors can have smaller values in this case.

Error-indicating Lights. Neon bulbs connected across the output terminals of each operational amplifier have long been found useful for



(b)

a 400-cps carrier. The output voltage range may be as large as from plus 200 volts to minus 200 volts with a load of 100,000 ohms.

indicating excessive output voltages and thus for eliminating computer runs during which one or more operational amplifiers were overloaded (see Sec. 7.2).

In the case of d-c amplifiers employing automatic balancing circuits, it is possible to use neon bulbs for indicating excessive signal voltages in the modulated carrier channel.[1] Such a neon bulb is shown connected to the output of the synchronous rectifier in the circuit diagram of Fig. 5.39. The circuit is arranged so that the neon "error-indicator light" will begin to glow whenever the absolute value of the voltage across the capacitor C_I exceeds 1 mv.

Assuming that the modulated carrier channel is working properly, the error-indicator light will glow whenever the feedback circuit constituting the operational amplifier is not acting to reduce the low-frequency error voltage on the capacitor C_I.

The error-indicator light will, thus, not simply indicate excessive output voltages in the operational amplifier in question. Output voltages substantially in excess of the threshold level of ordinary overload lights will, for instance, be tolerated if the amplifier is lightly loaded; this permits greater flexibility in the choice of scale factors and increased accuracy. Again, the new error-indicating system will indicate improper amplifier operation if an amplifier is loaded by too many potentiometers even though the output voltage may be relatively small. *Errors hitherto unnoticed can be eliminated in this manner, and more efficient utilization of the operational amplifiers is made possible.*

Finally, the error-indicating lights will tend to glow in certain cases of tube or component failures in the d-c amplifier channel. The new error indicators are seen to be more useful than ordinary overload lights; it may even be helpful to use both systems in conjunction so as to identify error sources more easily.

The RAND Automatic Balancing Circuit.[1] Figure 5.39 shows the automatic balancing and error-indicating circuits used by the RAND Corporation[2] in connection with a modified Reeves[3] d-c amplifier. The d-c amplifier is somewhat similar to that of the RCA circuit shown in Fig. 5.38*a*. An earlier version of the RAND automatic balancing circuit included a polystyrene grid-blocking capacitor of 1 μf with a grid leak resistor of 10 megohms. In order to avoid the necessity of discharging the grid-blocking capacitor after overloads, it was decided to use the cathode-follower input stage shown in Fig. 5.39; the low grid current (10^{-6} μa)

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[1] *Ibid.*

[2] Santa Monica, Calif.

[3] See Frost, S., Compact Analog Computer, *Electronics*, July, 1948, for the Reeves d-c amplifier circuit.

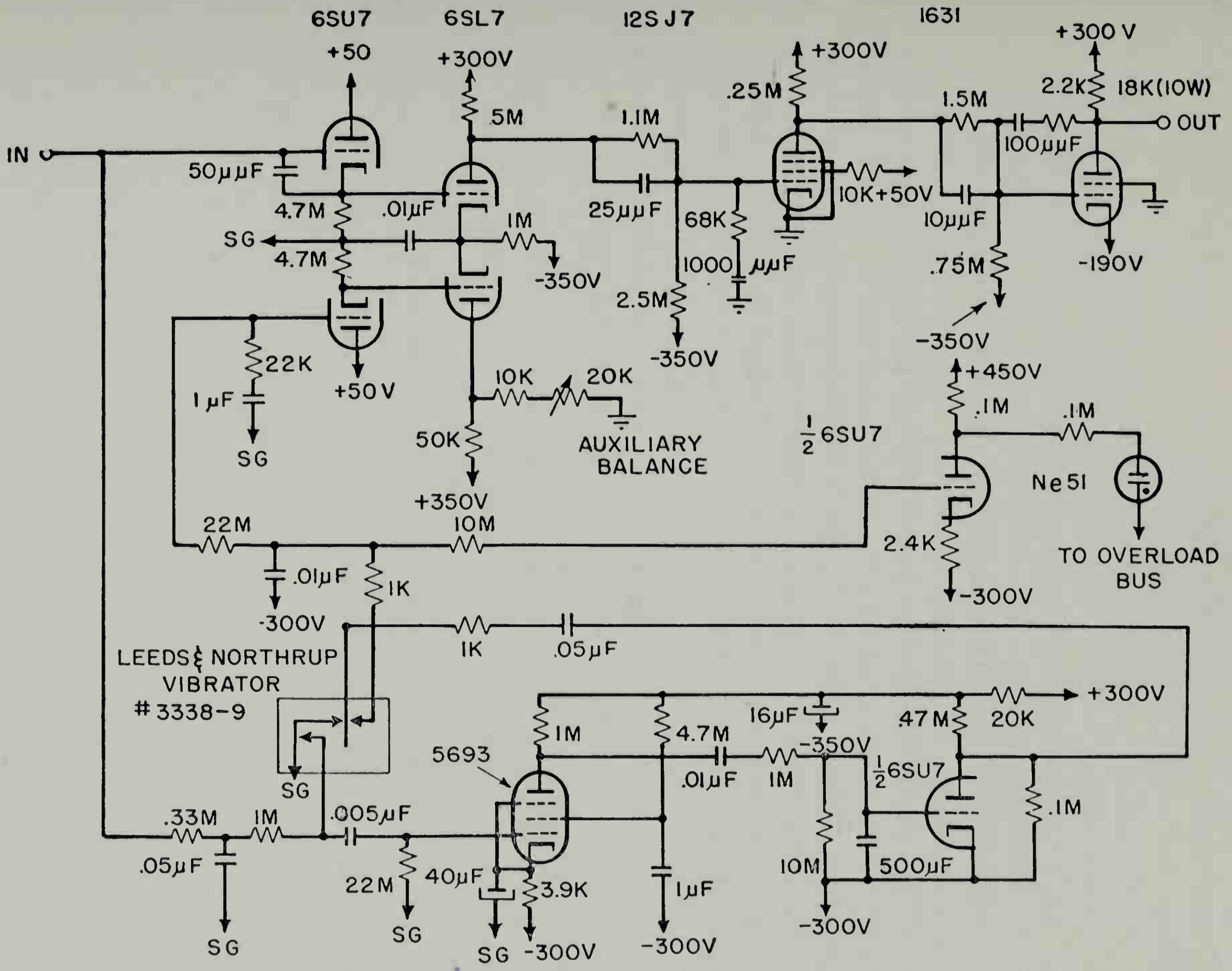

FIG. 5.39. High-quality d-c amplifier developed by the RAND Corporation, Santa Monica, Calif. The input circuits must be well shielded. Note the separate signal ground strap (SG) used to avoid circulating currents in the ground connections. The automatic balancing circuit incorporates an error-indicator light connected to an overload bus common to all amplifiers.

of this circuit makes the use of a grid-blocking capacitor practically unnecessary.

The circuit employs a 60-cps modulated carrier channel and the new error-indicating circuit. The vibrator contact arrangement results in a phase-sensitive peak detection scheme which is said to reduce the demodulator ripple filtering problem. An auxiliary balancing control is provided.

The synchronous vibrators used with automatic balancing circuits may present somewhat of a maintenance problem. Although many vibrators are capable of a useful life of more than 2,000 hr,[1] the life period decreases very sharply if the contacts are operated at open-circuit voltages in excess of the rated maximum voltage. In the RAND circuit, the second stage of the a-c amplifier has a purposely low plate voltage, so that excessive error voltages during overloads will tend to be limited and cannot damage the demodulator contacts of the vibrator. The error-indicating light appears at the output of a d-c amplifier stage following the synchronous rectifier. All error-indicating lights in the RAND computer are returned to a common bus; a warning buzzer or relay system can thus be actuated by an error indication in any amplifier.

The gain A_2 of the RAND modulated carrier channel is about 2,000, and the equivalent input d-c noise voltage is between 50 and 100 μv. A switch (not shown in Fig. 5.39) is provided to disconnect the automatic balancing circuit from the d-c amplifier after severe overloads.

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[1] Chance, Williams, Hughes, Sayre, and MacNichol, *op. cit.*

CHAPTER 6

MULTIPLICATION AND FUNCTION GENERATION

6.1. Miscellaneous Devices for the Multiplication of Two or More Variables. *Introduction.* Devices capable of multiplying two machine variables will generally be represented by a block diagram like the one shown in Fig. 6.1. The device will have two input voltages, X_1 and X_2, corresponding to the two machine variables to be multiplied. The output voltage X_o will be proportional to the product of the two input voltages X_1 and X_2 so that

$$X_o = cX_1X_2 \tag{6.1}$$

where c is a positive or negative constant.

The designer of operational amplifiers or computing networks (Chap. 4) can draw freely on the well-established theory of linear circuits. Such knowledge is, however, not sufficient for successful multiplier design. From the nonlinear nature of Eq. (6.1), it follows that *multiplication cannot be achieved by means of linear circuits alone.* In fact, all multipliers must involve devices whose gain is dependent upon one or more of the applied voltages.

FIG. 6.1. General block diagram of a multiplier for multiplying two d-c voltages.

At the very outset it must be said that the field here is still an open one. The design of the servomultipliers mentioned in Sec. 1.3 is fairly simple and well known to the art (see also Secs. 6.2 and 6.4). But the use of such electromechanical devices necessarily imposes limitations on the frequency ranges of some machine variables and thus on the computer time scale and the speed of computation. It would be very desirable to have purely electronic multiplying devices of fair accuracy (0.1 to 0.5 per cent of full output scale, or 0.001 to 0.005 machine unit) and as simple in design as a good d-c integrator (six or less vacuum tubes). Circuits satisfying both these requirements have not been developed as yet (1951). The portions of this chapter dealing with electronic multipliers are intended to show the directions in which the development of such circuits is proceeding rather than to instruct the reader in the construction of specific devices.

Multiplication of Positive and Negative Quantities. The ideal multiplier would accept voltages of both signs for either X_1 or X_2, and its output voltage X_o would have the correct sign associated with the product X_1X_2. Such a multiplier is said to be capable of *four-quadrant operation.* Other types are capable of *two-quadrant operation.* Here the input voltage X_1 may be either positive or negative, and the output voltage X_0 will change sign with the input voltage X_1. The input voltage X_2, however, can be only either positive or negative throughout the computation. Such multipliers are called *two-quadrant multipliers.* A third class of multipliers is capable of accepting only voltages of one sign for either X_1 or X_2. Such multipliers might be called *one-quadrant multipliers.*

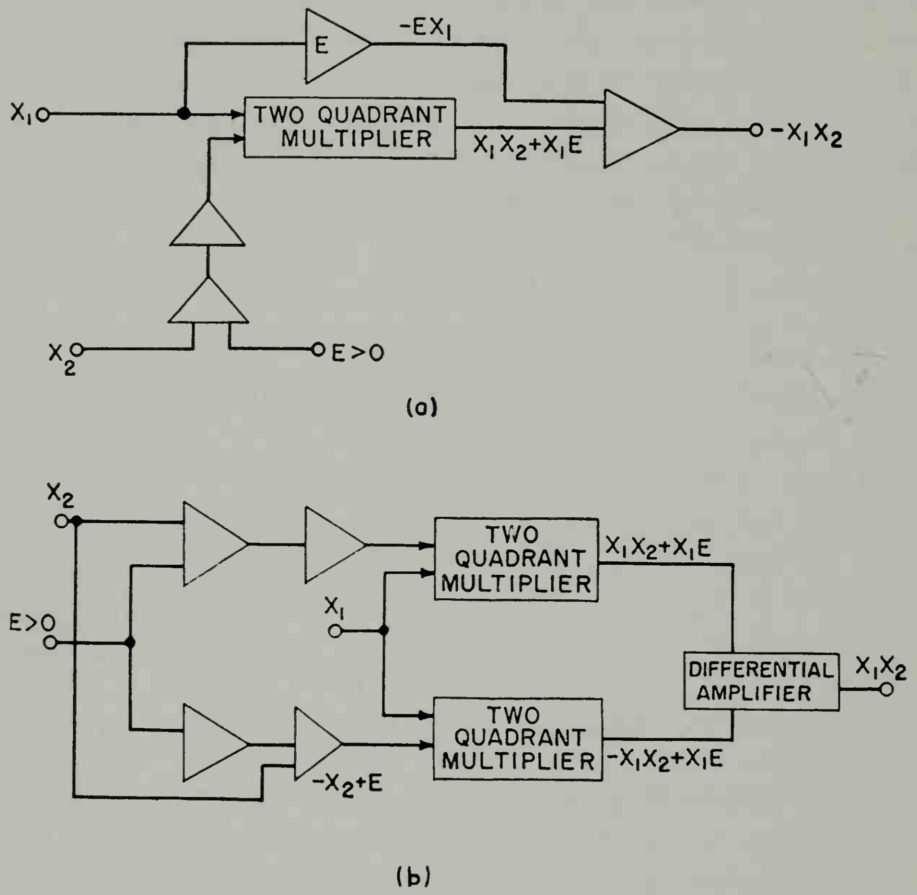


Fig. 6.2. Adaptation of two-quadrant multipliers to four-quadrant multiplication.

Generally speaking, it is usually possible to build four-quadrant multipliers by combining simpler multiplying devices which are not capable of four-quadrant operation. Figure 6.2*a* shows how a two-quadrant multiplier can be made to serve for four-quadrant multiplication. The multiplier is of a type which will accept positive and negative voltages X_1 but only values of the second input voltage, V, say, which are positive and greater than some value V_o. X_1 is to be multiplied by a machine variable

X_2 which can assume both positive and negative values. A constant positive voltage E, sufficiently large to make the sum of X_2 and E greater than V_o for all values of X_2, is added to the voltage X_2. The voltage

$$V = X_2 + E$$

is used as in input to the multiplier, which will compute the product

$$X_1V = X_1X_2 + X_1E$$

Figure 6.2*a* shows how the product X_1E may be subtracted from the multiplier output voltage in order to obtain the correct product X_1X_2. The arrangement of Fig. 6.2*b* utilizes two two-quadrant multipliers. This circuit may be more complex than that of Fig. 6.2*a*, but some errors may cancel because of the symmetrical push-pull arrangement.

Multiplying Circuits Using Special Function Generators. The nonlinear devices needed for multiplication may take the form of function generators whose output voltages are proportional to the square, logarithm, etc., of the respective input voltages. Thus

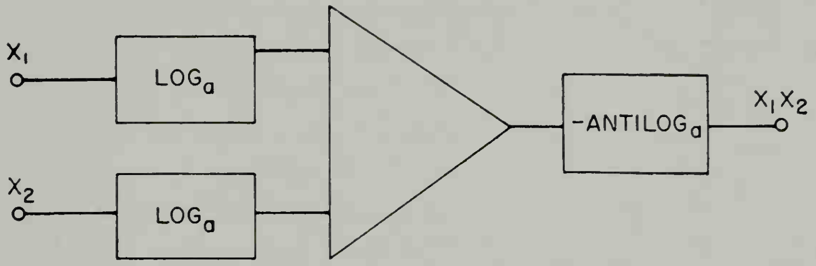


FIG. 6.3. Logarithmic multiplier.

$$X_1X_2 = a^{(\log_a X_1 + \log_a X_2)} = \text{antilog}_a\,(\log_a X_1 + \log_a X_2) \qquad (6.2)$$

Figure 6.3 shows the block diagram of a multiplier based on this well-known relation. It is easy to see that circuits based on the properties of the logarithmic function may be used to obtain division, powers, and roots as well as multiplication. Since the logarithmic function is not defined for negative values of the argument, the direct application of the circuit of Fig. 6.3 is restricted to positive values of X_1 and X_2 (one-quadrant operation).

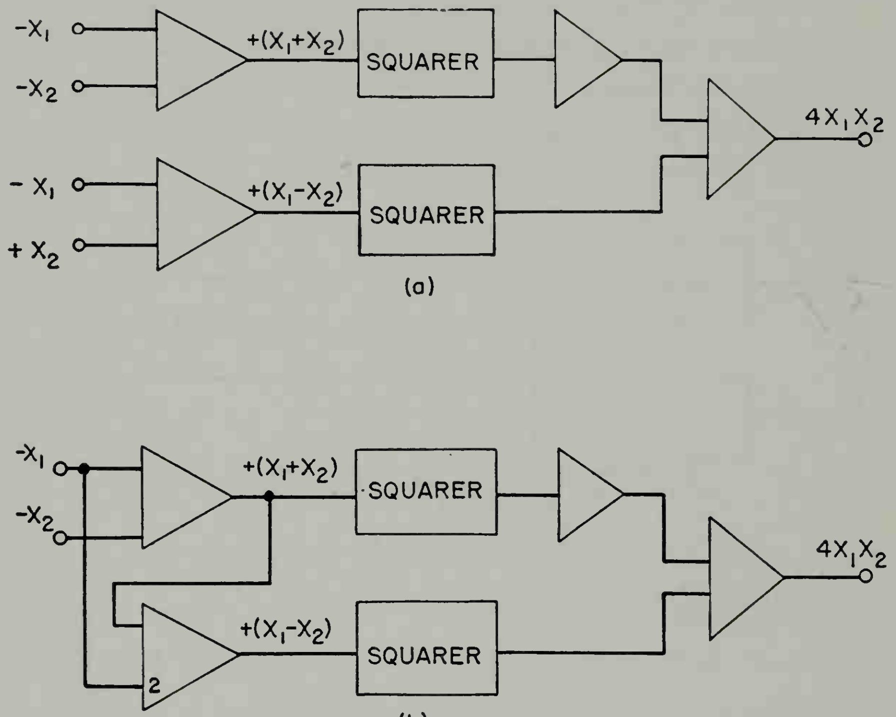
The *quarter-square multiplier* shown in Fig. 6.4*a* is based on the relation

$$X_1X_2 = \tfrac{1}{4}[(X_1 + X_2)^2 - (X_1 - X_2)^2] \qquad (6.3)$$

If squaring devices permitting negative as well as positive input voltages are available (*e.g.*, the photoformers described in Sec. 6.5), a quarter-square multiplier will permit direct multiplication of both positive and negative variables (four-quadrant operation). The percentage error in the output voltage of a quarter-square multiplier may be excessive for small values of the difference (6.3).

Many multiplying circuits based on similar principles can be devised; a number of function generators suitable for such applications are described in Secs. 6.5 and 6.7.

All electronic multipliers based on function generators may suffer from the difficulty of obtaining accurate all-electric function generators (see Sec. 6.5). It is clear that the more accurate servo-driven function generators will rarely be used in this connection, since servomechanisms are capable of direct multiplication.



(a)

(b)

FIG. 6.4. Quarter-square multipliers.

Voltage-sensitive Variable Gain Devices Used for Multiplication. The majority of devices used for the multiplication of two variables may be described by a block diagram such as the one shown in Fig. 6.5*a*. The gain or transfer function $K = X_o/X_1$ of the device in question with respect to an input voltage X_1 is variable and will increase or decrease with the value of a second input voltage X_2. Figure 6.5*b* and *c* shows two different specific examples of devices which can be described by the block diagram of Fig. 6.5*a*. Figure 6.5*b* shows the servo-positioned potentiometer already discussed in Sec. 1.3. The potentiometer gain or transfer function X_o/X_1 depends on the "gain-determining" voltage X_2 which determines the position of the potentiometer wiper. The output voltage X_o is, accordingly, a function of both input voltages X_1 and X_2 and may be made equal to the product X_1X_2 of the two input voltages. Figure 6.5*c* shows a variable-gain amplifier consisting of a multigrid vacuum tube with an input voltage X_1 applied to one control grid and a second input voltage X_2 applied to the second control grid. The output voltage X_o may be made roughly proportional to the product of these two input voltages.

There are several fundamental differences between the circuits shown in Fig. 6.5*b* and 6.5*c*, although the principle of operation as shown by the block diagram is the same in either case. The servo-driven potentiometer is an electromechanical device, whereas the variable-gain amplifier is a purely electronic device. Again, the output voltage of the potentiometer multiplier will be quite accurately proportional to the first input voltage X_1 unless there is excessive heating of the potentiometer winding. In the variable-gain amplifier, the output voltage will depend on a function of the input voltage X_1 which is only roughly linear; the output voltage is dependent on the tube characteristics.

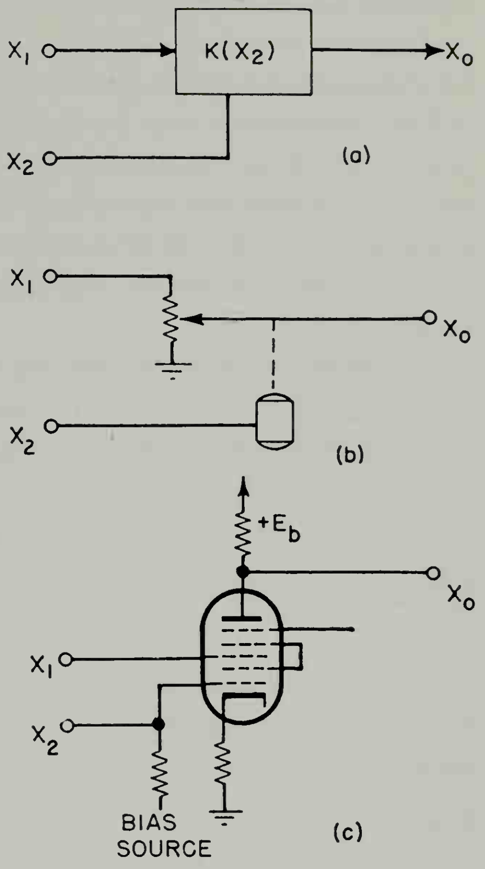


FIG. 6.5. Voltage-sensitive variable-gain devices used for multiplication of two d-c voltages.

Another difference between the two multiplying devices is in their respective abilities to multiply both positive and negative machine variables. The simple potentiometer of Fig. 6.5*b* is a two-quadrant multiplier. The variable-gain amplifier shown in Fig. 6.5*c* is a two-quadrant multiplier, since the input voltage X_1 may be either positive or negative if the bias levels are chosen correctly. It is seen, however, that a constant voltage depending on the plate supply voltage is added to the desired output voltage cX_1X_2. This zero-offset voltage may be subtracted out by one of the methods shown in Fig. 6.2. The variable-gain amplifier shown will, however, multiply with reasonable accuracy only for a very small range of operating conditions, and these operating conditions will be difficult to reproduce.

Multiplying Devices Based on the Use of Amplitude Modulation. A linear *amplitude modulator* is an electronic or electromechanical device capable of producing an output a-c voltage (modulated carrier) whose amplitude is proportional to a d-c input voltage. The modulator is said to be phase-sensitive if a change in the sign of the input voltage causes a 180-deg phase reversal of the a-c output voltage.

Carrier-controlled switch-type modulators utilize electronic or electromechanical switching devices which periodically interrupt the d-c input voltage to produce an a-c output voltage whose amplitude is accurately proportional to the d-c input voltage. Vacuum-tube switch modulators as well as the synchronous vibrators discussed in Sec. 5.6 are good examples. Switch-type modulators cannot be used directly for multiplication, since their output amplitude depends on the d-c input voltage alone.

Curvature modulators depend on the characteristics of nonlinear circuit elements, such as vacuum tubes or rectifiers, to "mix" the input d-c voltage with an input a-c "carrier" voltage. Suppose the sum of a d-c voltage

$$V_1 = X_1 \tag{6.4}$$

and an a-c voltage

$$V_2 = X_2 \sin \omega_c \tau \tag{6.5}$$

is applied to a nonlinear device; ω_c is the carrier circular frequency. For the sake of simplicity, consider a square-law detector with the output voltage

$$\begin{aligned} k(V_1 + V_2)^2 &= k(X_1^2 + X_2^2 \sin^2 \omega_c \tau + 2X_1X_2 \sin \omega_c \tau) \\ &= k\left(X_1^2 + \frac{1}{2} X_2^2\right) - \frac{k}{2} X_2^2 \cos 2\omega_c \tau + 2kX_1X_2 \sin \omega_c \tau \qquad (6.6) \end{aligned}$$

where k is a constant. The last term on the right is seen to be proportional to the product of the two input voltages (6.4) and (6.5) and constitutes the desired modulated a-c output voltage. The d-c and second harmonic terms in Eq. (6.6) can be eliminated from the output voltage by means of blocking capacitors and push-pull circuits[1] (so-called balanced modulators).

Figure 6.6 shows the block diagram of a multiplier based on the use of amplitude modulation. A switch-type multiplier is used to produce a modulated carrier whose amplitude is proportional to X_2. The amplitude of this modulated carrier is then multiplied by the d-c voltage X_1 through the use of a curvature multiplier. The latter produces an a-c voltage whose amplitude is proportional to the product X_1X_2. This modulating signal is then "detected" by a phase-sensitive demodulator[1] whose output voltage is a d-c voltage proportional to X_1X_2. The possible utilization of *saturable transformer modulators*[2] (*magnetic amplifiers*) in the block diagram of Fig. 6.6 might deserve further investigation.

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[1] Detailed discussions of many practical modulator circuits are presented in B. Chance, F. C. Williams, V. Hughes, D. Sayre, and E. F. MacNichol, *Waveforms*, Chap. 11, MIT Radiation Laboratory Series, Vol. 19, McGraw-Hill, New York, 1949.

Multipliers based on amplitude modulation are capable of good frequency response if a sufficiently high carrier frequency is chosen. The accuracy of such devices must depend essentially on the specific nonlinear characteristics of the curvature modulator circuit used and may be between 1 and 8 per cent. A possible method for improving the accuracy by means of feedback is discussed in Sec. 6.2.

Dynamometer-type Multipliers. An electrodynamometer consists essentially of a moving coil, somewhat like those used in meter movements, which can rotate in the field of an electromagnet as shown in Fig. 6.7. If the direct current in the coil is i_1 and the direct current in the electromagnet is i_2, the resulting torque on the moving coil will be proportional to the product i_1i_2 of the two currents i_1 and i_2.

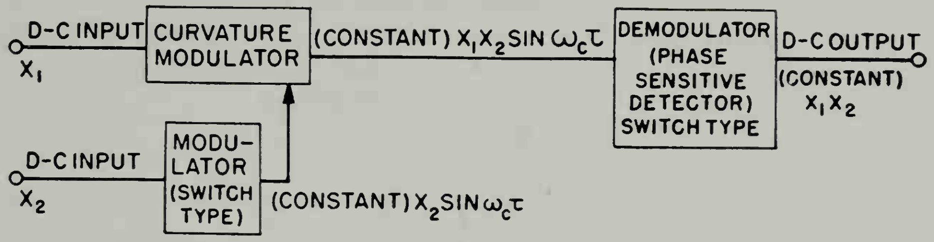

D-C INPUT
X_1
CURVATURE MODULATOR
(CONSTANT) $X_1X_2 \sin \omega_c \tau$
DEMODULATOR (PHASE SENSITIVE DETECTOR) SWITCH TYPE
D-C OUTPUT
(CONSTANT) X_1X_2
D-C INPUT
X_2
MODULATOR (SWITCH TYPE)
(CONSTANT) $X_2 \sin \omega_c \tau$

FIG. 6.6. Block diagram of a multiplier based on amplitude modulation.

If two electrodynamometers are arranged with their moving coils rigidly connected to a common shaft as in Fig. 6.7, the resulting torque will be zero if and only if

$$i_1i_2 = -i_3i_4 \tag{6.7}$$

where i_1, i_2 and i_3, i_4 are the currents in the first and second electrodynamometers, respectively. This balance condition may be maintained by using an electromechanical or photoelectric pickoff to convert any angular displacement of the dynamometer shaft into an error voltage; the latter is then used to drive a high-gain current amplifier whose output is equal to i_3 (see Fig. 6.7). If the polarity of the error voltage is chosen correctly, such a feedback arrangement will tend to minimize the shaft displacement and thus the shaft torque at least for slow variations of the currents i_1 and i_2. As a result, one has very nearly

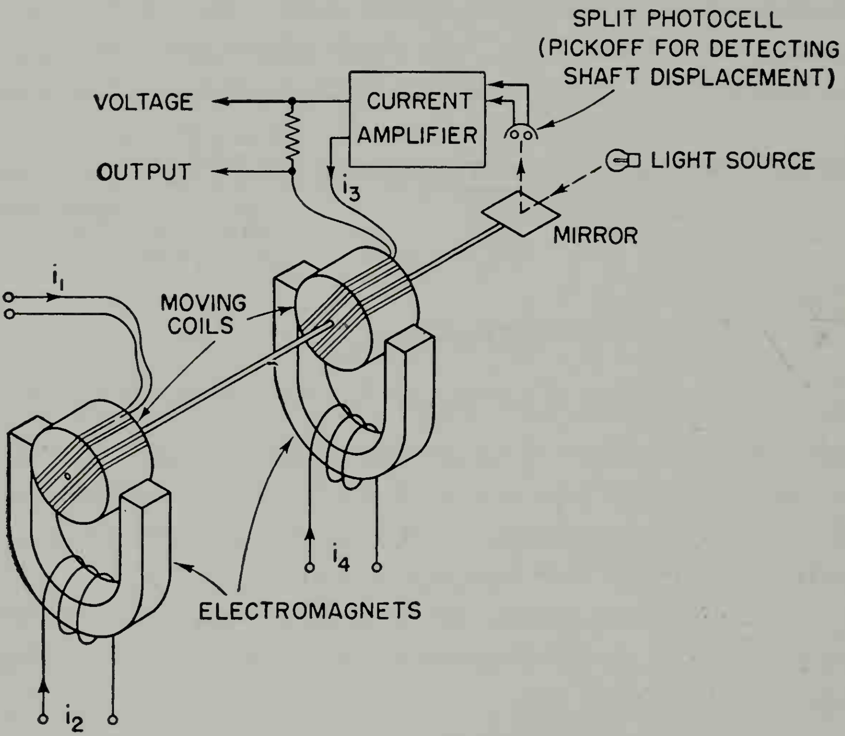
$$i_3 = -\frac{i_1i_2}{i_4} \tag{6.8}$$

---

[1] Practical demodulator circuits are discussed in detail in *ibid.*, Chap. 14.

[2] *Ibid.*

for a certain range of working frequencies for i_1 and i_2. The relation (6.8) can be made the basis of a compact d-c voltage multiplier if current amplifiers, which produce currents proportional to d-c input voltages, are available. If the current i_4 is made proportional to a third input voltage, simultaneous division and multiplication is possible. Accuracies better than 2 per cent should be possible. The frequency response of the electrodynamometer multiplier is limited to perhaps 100 cps by the effects of the inertia of the moving coils and of coil inductance; a detailed analysis of these effects must be made in each case.



· Fig. 6.7. Principle of the electrodynamometer multiplier.

At least one multiplying device operating on a somewhat similar principle has been built commercially.[1] The accuracy of this commercial multiplier is between 1 and 2 per cent at signal frequencies up to 5 cps.

The Crossed-fields Electron-beam Multiplier.[2] The principle of the crossed-fields electron-beam multiplier is somewhat similar to that of the electrodynamometer multiplier. Unlike the latter, the new multiplier does not contain any moving parts, so that its high-frequency response is excellent.

Figure 6.8 illustrates the principle of the crossed-fields multiplier. An ordinary cathode-ray tube having electrostatic deflection plates and a

[1] Specialties, Inc., Skunk's Misery Road, Syosset, Long Island, New York.

[2] MacNee, A. B., An Electronic Differential Analyzer, *Proc. IRE*, **37**: 1315, 1949.

short persistence screen is fitted with an axial coil wound around the tube in the vicinity of the second pair of deflection plates.

Let E_1 and E_3 be the voltages across the first and second pair of deflection plates, respectively, and let i_2 be the current in the axial coil. The first pair of deflection plates shown will impart a vertical velocity proportional to E_1 to the electron beam. As a result, the beam experiences a *horizontal* magnetic deflection force proportional to $E_1 i_2$ as the electrons pass through the coil.

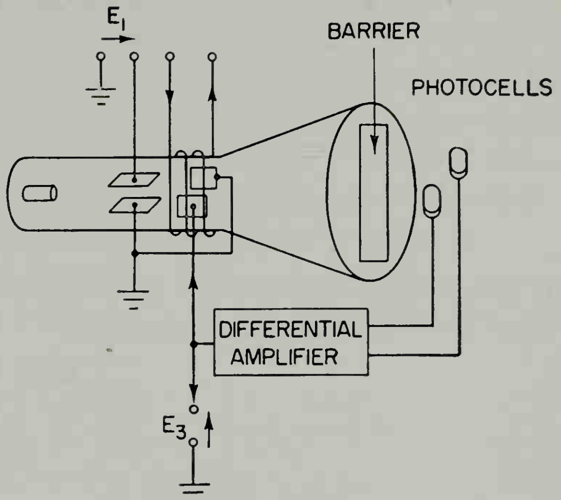


Fig. 6.8. Principle of the crossed-fields electron-beam multiplier (A. B. MacNee, *Proc. IRE*, **37**: 1315, 1949.)

Any actual deflection of the electron beam will be detected by a pickoff consisting of a pair of high-gain photoelectric cells arranged on either side of a vertical barrier in front of the cathode-ray tube screen.

The difference between the photocell output voltages is amplified by a differential amplifier. The amplifier output voltage E_3 is applied to the second pair of deflection plates in such a manner as to counteract the original deflection. The plates will exert a *horizontal* force proportional to the voltage E_3 on the electron beam. If the gain of the feedback loop consisting of the cathode-ray tube, the photocells, and the amplifier is large, the horizontal deflection of the electron beam will be kept very small. The horizontal forces acting on the beam must then balance each other, so that at least approximately

$$E_3 = (\text{constant})\, E_1 i_2 \tag{6.9}$$

If E_1 and i_2 are made proportional to two input voltages X_1 and X_2, the output voltage will be proportional to the product $X_1 X_2$. Equation (6.9) compares favorably with the experimentally determined performance equation

$$E_3 = (\text{constant})\, (E_1 i_2 + 0.009 E_1^2 + 0.015 i_2 + 0.0005) \tag{6.10}$$

of a typical unit. It is believed that the error terms could be reduced by a factor of 10 through better magnetic shielding and through careful positioning of the photocells and the barrier. The use of two units in a push-pull circuit and of automatic balancing devices (see Sec. 5.6) might result in further improvements.

The crossed-fields electron-beam multiplier is a useful and fairly simple device. Perhaps its greatest advantage is its good high-frequency response, which makes this multiplier especially useful for repetitive computers. The frequency response is flat (within 1 per cent) and the phase shift negligible up to *at least* 20,000 cps for E_1 and 3,000 cps for i_2; the latter figure is capable of improvement through suitable equalization of the current amplifier driving the axial coil.

Finally, if the connections in Fig. 6.8 are changed so that the first pair of deflection plates receives its deflection voltage E_1 from the amplifier whereas an input voltage E_3 is applied across the second pair, then

$$E_1 = (\text{constant}) \frac{E_3}{i_2} \qquad i_2 \neq 0 \tag{6.11}$$

is the output voltage; the crossed-fields electron-beam multiplier may be used for *division* in this manner.

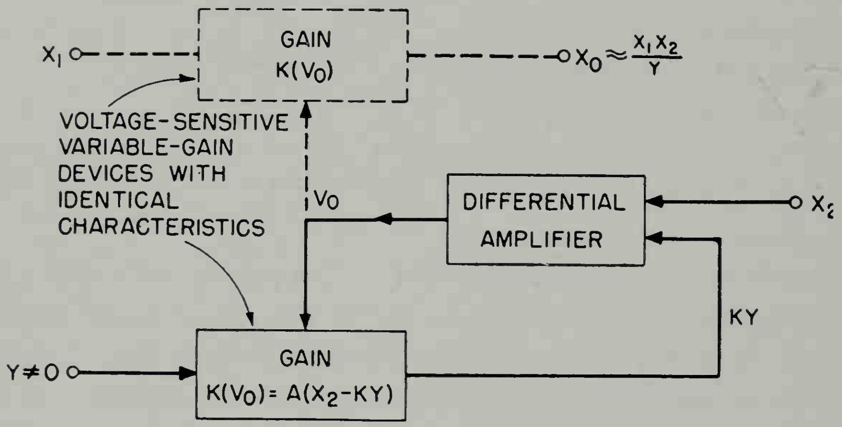


FIG. 6.9. Feedback circuit used for setting the gain of a variable-gain device accurately to a value determined by the input voltages X_2 and Y.

6.2. Multiplying Devices Using Feedback to Linearize the Gain Characteristics. *Introduction.* Frequently, the gain K of a voltage-sensitive variable-gain device can be made *accurately* proportional to the ratio X_2/Y of two voltages (machine variables) X_2 and Y through the use of a *feedback circuit.* Figure 6.9 shows such an arrangement. The gain K of the variable-gain device shown in the solid lines *must be independent of its input voltage* (Y in this case). The gain K does vary monotonically but not necessarily linearly with the gain-setting voltage V_o obtained at the output of a high-gain differential amplifier. The latter is seen to amplify the difference between the input voltage X_2 and the voltage YK. Since K varies monotonically with V_o, it is possible to write

$$K = K(V_o) = A(X_2 - YK) \qquad Y \neq 0 \tag{6.12}$$

where A is a positive quantity proportional to the amplifier gain but in general depending on the value of V_o. It follows that

$$K = \frac{AX_2}{1 + AY} \tag{6.13}$$

If the value of A is made sufficiently large by increasing the amplifier gain, Eq. (6.13) will reduce to

$$K \approx \frac{X_2}{Y} \tag{6.14}$$

The feedback will thus tend to set the gain K automatically near the value X_2/Y. The accuracy with which Eq. (6.14) approximates the correct expression (6.13) will depend on the value of the amplifier gain.[1]

Feedback Multipliers. Assume, now, that it is possible to obtain a second variable-gain device whose gain will vary with the voltage V_o exactly like the gain of the first variable-gain device.

If, then, the output voltage V_o of the differential amplifier shown in Fig. 6.9 is fed to the second variable-gain device, the latter can be used to multiply any input voltage X_1 by $K \approx X_2/Y$. The dotted lines in Fig. 6.9 indicate how the multiplication

$$X_o = X_1 \frac{X_2}{Y} \tag{6.15}$$

may be performed in this manner. It is seen that simultaneous multiplication and division (see Sec. 6.3) is possible by this method. One may perform several multiplications by X_2/Y by providing additional variable-gain devices with identical characteristics and with gains determined by the voltage V_o.

The reader will find it interesting to make an analysis of the effects of unbalance voltages due to drift at different points of the circuit in the manner of Sec. 5.1.

Example: Computer Servomechanisms. The operation of the ordinary *multiplying servomechanisms* (which have been shown only as block-diagram symbols in the previous sections) depends on this principle. The variable-gain devices in this case are potentiometers positioned by a servomotor which is actuated by the output voltage V_o of a differential amplifier. The latter may take the form of a servo amplifier or of a sensitive relay. Figure 6.10 shows such an arrangement; the servomechanism shown is a four-quadrant multiplier. The actual construction of practical computer servomechanisms is discussed in Sec. 6.4.

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[1] A uniformly high gain can be obtained only for a limited range of signal frequencies in each case, and the entire circuit will have to be carefully designed for high-frequency stability in a manner similar to that indicated in Sec. 5.4.

Design Problems. Aside from certain difficulties involved in attaining stable feedback action throughout the desired range of operating frequencies, *the chief problems in designing multipliers based on the block diagram of Fig.* 6.9 *are in obtaining*

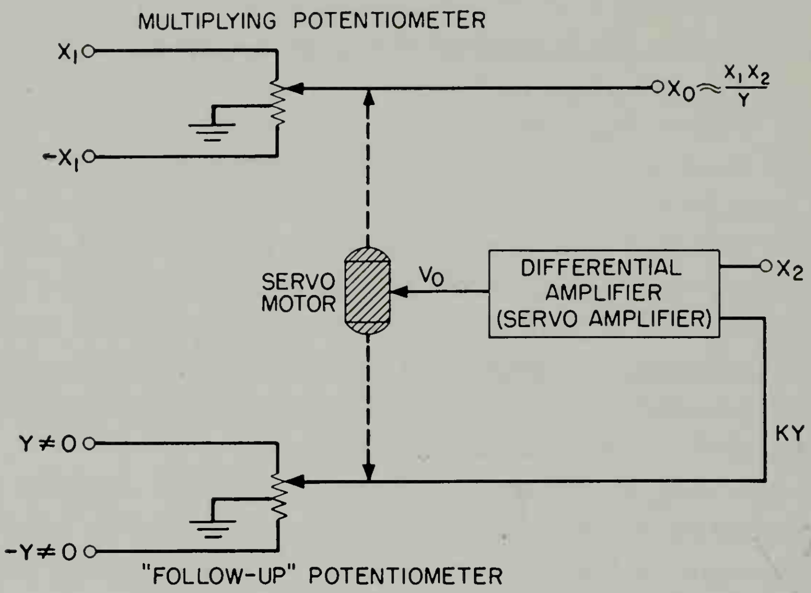
MULTIPLYING POTENTIOMETER
X_1
$-X_1$
$X_0 \approx \frac{X_1 X_2}{Y}$
SERVO MOTOR
V_0
DIFFERENTIAL AMPLIFIER (SERVO AMPLIFIER)
X_2
KY
$Y \neq 0$
$-Y \neq 0$
"FOLLOW-UP" POTENTIOMETER

FIG. 6.10. Principle of computer servomechanisms.

1. *Voltage-sensitive variable-gain devices whose gain is independent of the input voltage* (X_1 or Y in Fig. 6.9)
2. *Two or more voltage-sensitive variable-gain devices whose gain vs. gain-setting voltage characteristics are accurately similar*

Servo-driven potentiometers, such as those used in the circuit of Fig. 6.10, satisfy the last two requirements quite well and permit multiplications (and divisions; see Sec. 6.3) with errors lower than 0.2 per cent of full scale. The use of electromechanical devices of this type will, however, limit the range of operating frequencies for the variables X_2 and Y in Fig. 6.10 to the range between 0 and about 15 cps.

Practically all simple voltage-sensitive circuit elements, such as thyrite resistors, crystals, or even vacuum tubes, do not satisfy either of the conditions given above; their gain may be dependent on the input voltage, and their characteristics cannot be reproduced with sufficient accuracy.

In certain types of variable-gain devices, the effect of the vacuum-tube characteristics can be made less important by using the tubes merely as *switches* (on-off devices). Whenever tubes are used in this manner, the circuit gain will be independent of the tube characteristics to a first-order approximation. This principle is applied in pulsed attenuator multipliers.

Pulsed Attenuator Multipliers.[1] Figure 6.11 illustrates the principle of the *pulsed attenuator.* The switch shown will attenuate the input signal X_1 in the ratio $K:1$ by interrupting (grounding) it periodically for a time interval t_{off} per period, with a repetition frequency which must be large compared with the highest signal frequency. The output voltage is smoothed by an RC filter. The transfer function K of the attenuator is proportional to

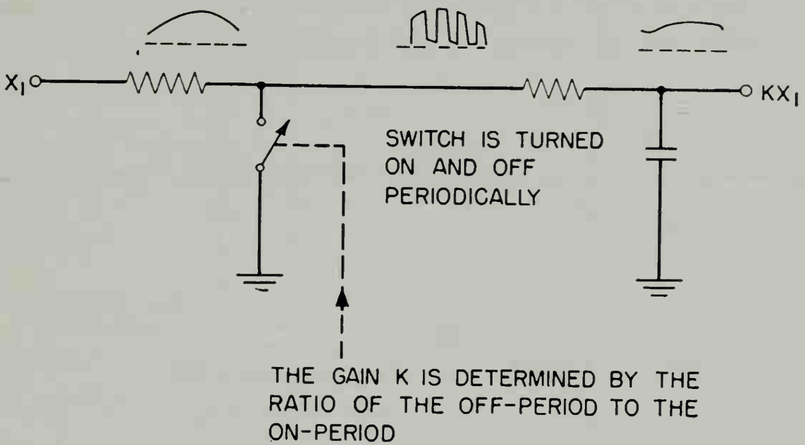
X₁
KX₁
SWITCH IS TURNED ON AND OFF PERIODICALLY
THE GAIN K IS DETERMINED BY THE RATIO OF THE OFF-PERIOD TO THE ON-PERIOD

FIG. 6.11. Principle of the pulsed attenuator.

$$1 - nt_{\text{off}}$$

In practice, the repetition frequency n is held constant, and the off-period t_{off} is varied in accordance with an applied voltage, so that a voltage-sensitive attenuator results.

Figure 6.12*a* shows two pulsed attenuators arranged as a feedback multiplier according to the block diagram of Fig. 6.9. Type 6V6 tubes are used for switching. Each switching tube is normally cut off by a negative bias voltage but is made conducting periodically by strong positive pulses from a *delay multivibrator.*[2] The latter is triggered at a constant frequency n by means of pulses from a master oscillator. *The period t_{off} of the positive pulses, and thus the transfer function K of each attenuator, will vary with the output voltage V_o of the differential amplifier shown.* As a result of the feedback through this amplifier and one of the attenuators, the gain K of each attenuator will become practically equal to X_2/Y, just as in Fig. 6.9.

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[1] Cornell thesis of A. C. Beer and H. W. Boehmer. Unfortunately, this thesis is not generally available to the public.

[2] For a detailed discussion of delay multivibrators, see Chance, Williams, Hughes, Sayre, and MacNichol, *op. cit.*, Chap. 5.

The complete circuit of a pulsed attenuator multiplier is shown in Vol. 21 of the MIT Radiation Laboratory Series.[1] Since the switching tubes shown in Fig. 6.12*a* require a positive plate voltage to operate, a positive voltage proportional to the plate voltage E_b must be added to X_1 unless the latter exceeds about 25 volts. The output voltage in Fig. 6.12*a* will, then, not be proportional to X_1X_2/Y, but will contain a constant multiple of X_2/Y which must be subtracted out in the manner of Fig. 6.2. The push-pull circuits shown in Fig. 6.12*b* and *c* are useful for this purpose and will also improve the accuracy of the computation somewhat.

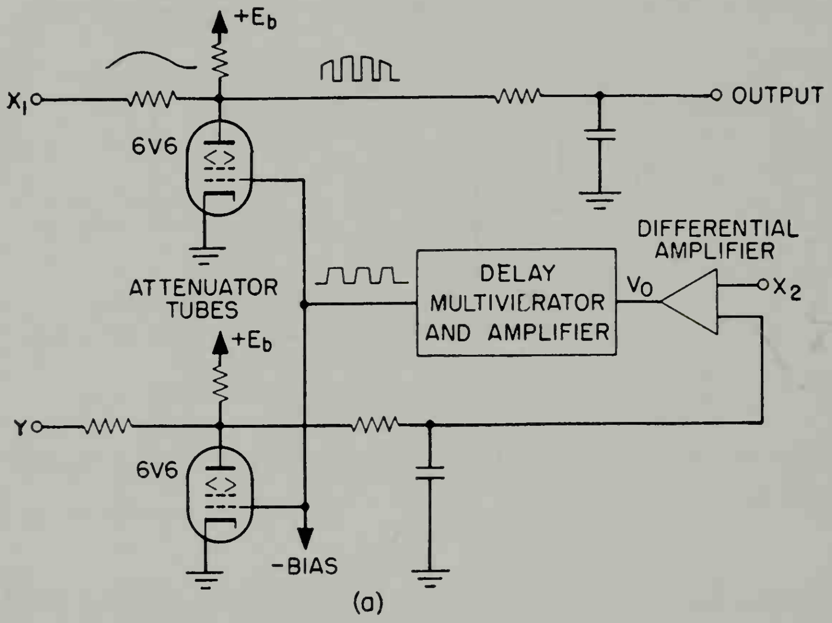

Fig. 6.12*a*. Two pulsed attenuators arranged as a feedback multiplier.

By careful matching of the switching tubes used in the two attenuators of a multiplier and by judicious application of d-c amplifier techniques (see Chap. 5) for avoiding drift, accuracies better than 0.2 per cent of full scale may be obtained with pulsed attenuator multipliers. These all-electronic devices may be used at signal frequencies well above 2,000 cps. The chief drawback is the relative complexity of the pulsed attenuator circuits; between seven and ten tube envelopes are required for a complete multiplier. If more than one variable is to be multiplied by X_2/Y, one feedback and comparison circuit will, however, serve for any number of such multiplications.

The Vibration Multiplier. If a condenser microphone[2] is driven at a constant audio frequency by means of a loudspeaker, the amplitude of

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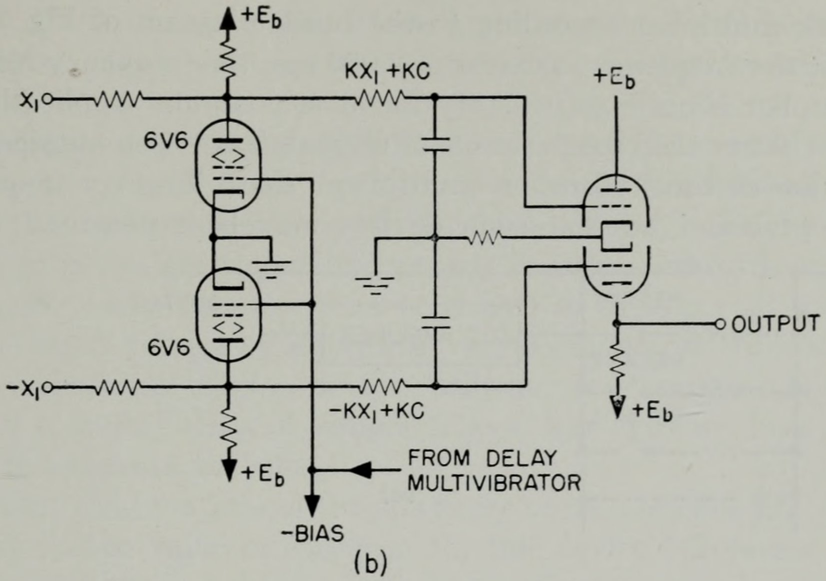
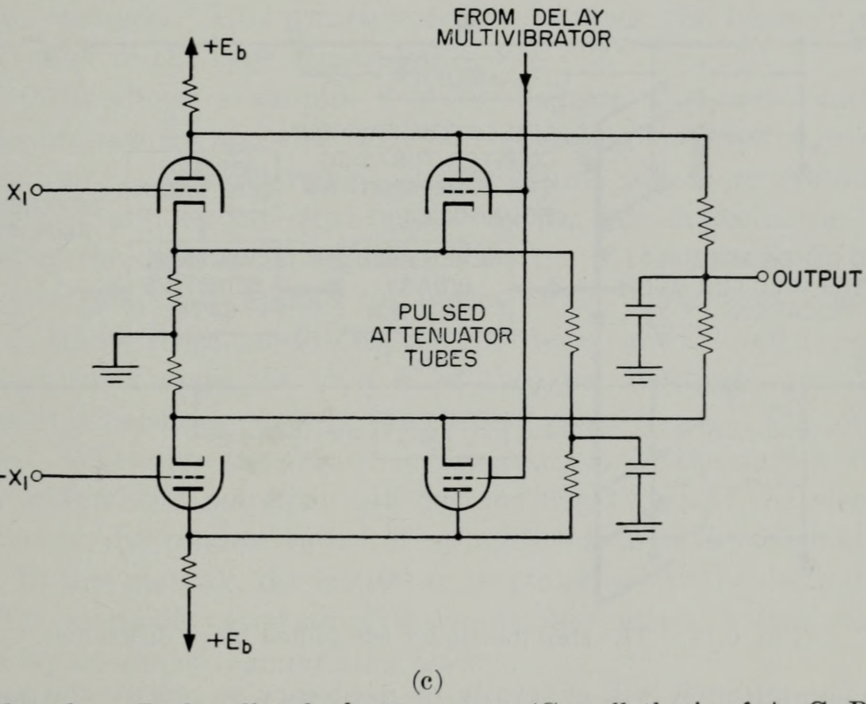
[1] Greenwood, I. A., J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948.

[2] See J. G. Frayne and H. Wolfe, *Elements of Sound Recording*, Wiley, New York, 1949, for a discussion of condenser microphones.





(c)

FIG. 6.12*b* and *c*. Push-pull pulsed attenuators (Cornell thesis of A. C. Beer and H. W. Boehmer, under the direction of H. S. Sack). Two attenuators are required for every multiplier.

the resulting a-c microphone output voltage is accurately proportional to the d-c voltage across the microphone plates. The output voltage amplitude is also proportional to a monotonic function of the amplitude of the loudspeaker excitation voltage. The device is thus an electromechanical modulator. Two identical microphone elements may be placed between a single set of condenser plates and driven by a single loudspeaker to form a feedback multiplier according to the block diagram of Fig. 6.9. By using a carrier frequency in excess of 2,000 cps, the frequency response of this multiplier is quite satisfactory for most computer applications, and accuracies better than 0.3 per cent of full scale have been obtained. The construction of the "vibration multiplier" does, however, require considerable precision, so that such devices may be expensive. In a d-c analog computer, it will generally be necessary to rectify the a-c output voltage, and some drift will result from this operation.

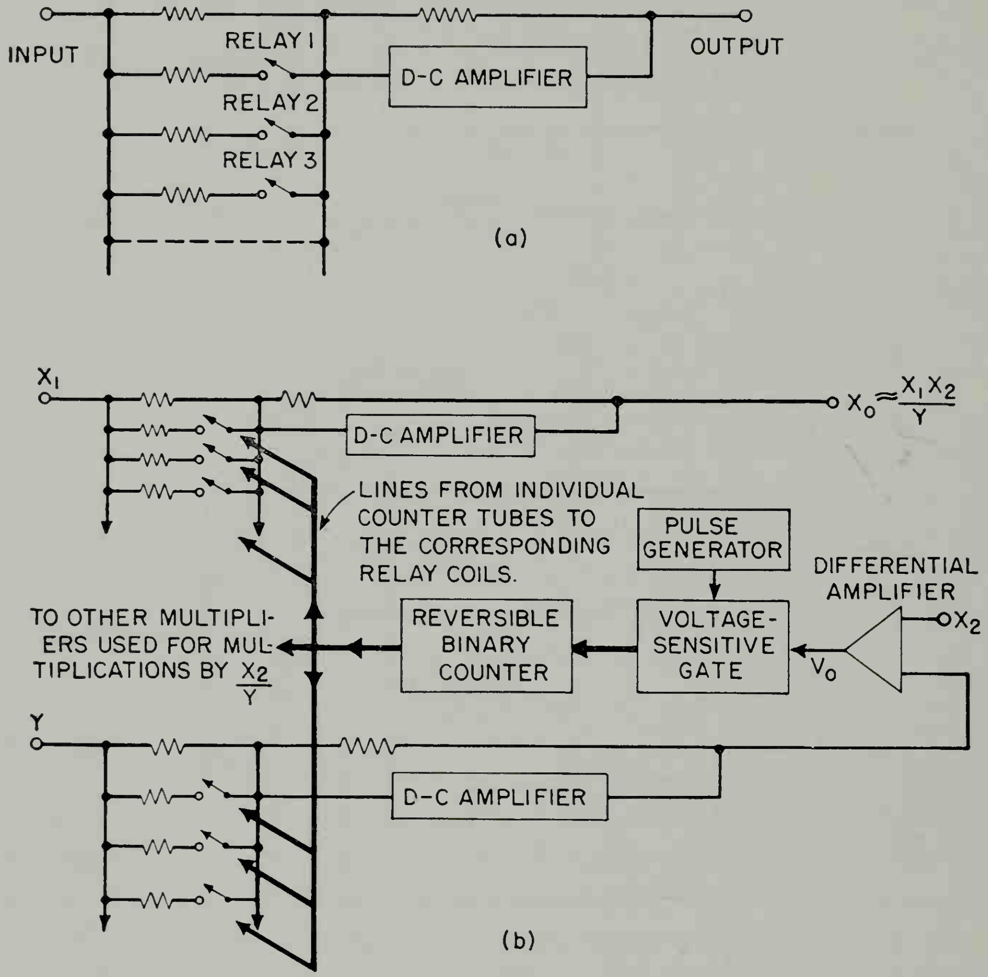

INPUT
RELAY 1
RELAY 2
RELAY 3
D-C AMPLIFIER
OUTPUT
(a)
X_1
D-C AMPLIFIER
$X_0 \approx \frac{X_1 X_2}{Y}$
LINES FROM INDIVIDUAL COUNTER TUBES TO THE CORRESPONDING RELAY COILS.
PULSE GENERATOR
DIFFERENTIAL AMPLIFIER
X_2
TO OTHER MULTIPLIERS USED FOR MULTIPLICATIONS BY $\frac{X_2}{Y}$
REVERSIBLE BINARY COUNTER
VOLTAGE-SENSITIVE GATE
V_0
Y
D-C AMPLIFIER
(b)

FIG. 6.13. The step multiplier (simplified block diagrams).

The Step Multiplier. The step multiplier, developed by the RCA Research Laboratories,[1] is a new, accurate computing element useful for multiplications by functions of machine variables as well as for multiplications by machine variables. The step multiplier is not a pure analog computing element since it uses a digital counter to regulate the gain of a variable-gain amplifier.

Figure 6.13 illustrates the principle of the step multiplier. Figure 6.13*a* shows an operational amplifier whose input resistance can be varied by placing different combinations of resistors in parallel by means of a set of relays.

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[1] Princeton, N.J.

If the relays are initially open and are closed successively as time progresses, the gain K of the operational amplifier will increase progressively as the input resistance is effectively decreased. If different combinations of relays are closed (or opened) in succession with equal time intervals, the various resistors can be chosen so as to permit multiplication of the input voltage by a stepwise varying monotonic function of the time. A somewhat similar device, utilizing one large stepping relay instead of a multiplicity of simple relays, has actually been used to generate functions of the time.[1]

If different combinations of the relays are opened or closed in a manner dependent on the value of a *voltage* V_o, this device becomes a voltage-sensitive variable-gain amplifier whose gain K varies step by step as the voltage V_o changes. This principle has been made the basis of a feedback multiplier of the type illustrated in Fig. 6.9.

Figure 6.13*b* shows a simplified block diagram of a step multiplier. The relays are opened and closed by corresponding stages of a reversible binary counter.[2] The reversible counter counts pulses generated by a pulse generator and is controlled (gated) by the gain-determining output voltage V_o of the differential amplifier shown. If the voltage V_o is positive, the electronic gate applies the pulses to the counter so that it will count up. Under these conditions, the counter stages will open and close relays so as to increase the gain K of the two identical variable-gain devices until it becomes equal to the correct value X_2/Y. The gate will then close. Whenever the gain K is less than X_2/Y, the output voltage V_o of the differential amplifier will become negative and the electronic gate will cause the counter to count up and thus increase the multiplier gain K. In this manner, the latter can be stabilized at the desired value X_2/Y. The stepwise variation of the multiplier output voltage may be smoothed by means of a smoothing filter.

The accuracy of the step multiplier can be improved by increasing the number of relays used. With 14 counter stages, the tubes of the binary counter could energize the relays in the sequence (none); (1); (2); (1,2); (3); (1,3); (2,3); (1,2,3); (4); etc.; the 28 possible relay positions permit $2^{14} = 16{,}384$ distinct values of the multiplier gain or a theoretical accuracy of better than 0.01 per cent of full scale. The gain can be made to increase or decrease in equal steps if the resistance values of the resistors associated with the different relays are chosen judiciously. The actual accuracy of the step multiplier is limited to about 0.02 per cent by d-c amplifier drift and by the accuracy of the gain-setting resistors.

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[1] Hagelbarger, D. W., Howe, C. E., and R. M. Howe, *Investigation of the Utility of an Electronic Analog Computer in Engineering Problems*, UMM-28, Apr. 1, 1949, Aeronautical Research Center, University of Michigan, Willow Run, Mich.

[2] For a detailed discussion of the circuits used, see *Project TYPHOON Progress Rept.* 2, Contract N6oni196, RCA Laboratories Division, Princeton, N.J.

The frequency range of the gain-setting variable X_2 in step multipliers is severely limited by the frequency response of the relays. Not only must the relays open or close within the time required for the smallest gain changes under consideration, but the relays in two or more variable-gain devices must "track" in order to yield accurately similar gain vs. voltage characteristics. The maximum counting rate usable with high speed relays is of the order of 1,500 counts per second, corresponding to a maximum voltage rate of change

$$\frac{dX_2}{d\tau} \leq \frac{1{,}500\,V_{\max}}{2^n} \tag{6.16}$$

where $V_{\max}$ is the voltage range and n is the number of relays used. Generally speaking, step multipliers can be about as fast as servomechanisms of equal accuracy. The accuracy can be improved at the expense of frequency response.

The step multiplier may be used as an accurate function generator if the resistors associated with the various relays are chosen correctly. Although one counter and gate circuit may, in any case, be used for several multiplications by the same machine variable or by functions of this machine variable, the relative complexity of the step multiplier circuits (over 50 tubes may be required for a single 14-relay multiplier) will restrict the usefulness of such devices to applications where high accuracy is of paramount importance.

Time-sharing Schemes. The very serious difficulty in obtaining two or more identical attenuator elements for use in feedback multipliers can sometimes be resolved by using a single attenuator on a time-sharing basis. As an example, Fig. 6.14 shows a *commutated-potentiometer servo* of the type developed at the RCA Research Laboratories.[1] The circuit shown is essentially the same as the servomultiplier of Fig. 6.10. *A single potentiometer is switched back and forth between the multiplying and gain-adjusting sections of the circuit.* This is accomplished by means of three commutator-type switches rotated at 3,600 rpm or by means of synchronous vibrators. The potentiometer output voltage for each switch position is "remembered" by an *RC* smoothing network which filters out the commutator ripple. The use of a single potentiometer in a computer servomechanism assures perfect "tracking" of the multiplying and follow-up sections of the circuit; no expensive accurately linear wire-wound potentiometers are required. A fairly low-cost potentiometer having good resolution, such as a metalized or carbon type, may be used. The frequency response of the servomechanism is not affected by the new arrangement and still constitutes a limit to the usefulness of the multiplier.

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[1] Princeton, N.J. (TYPHOON Project).

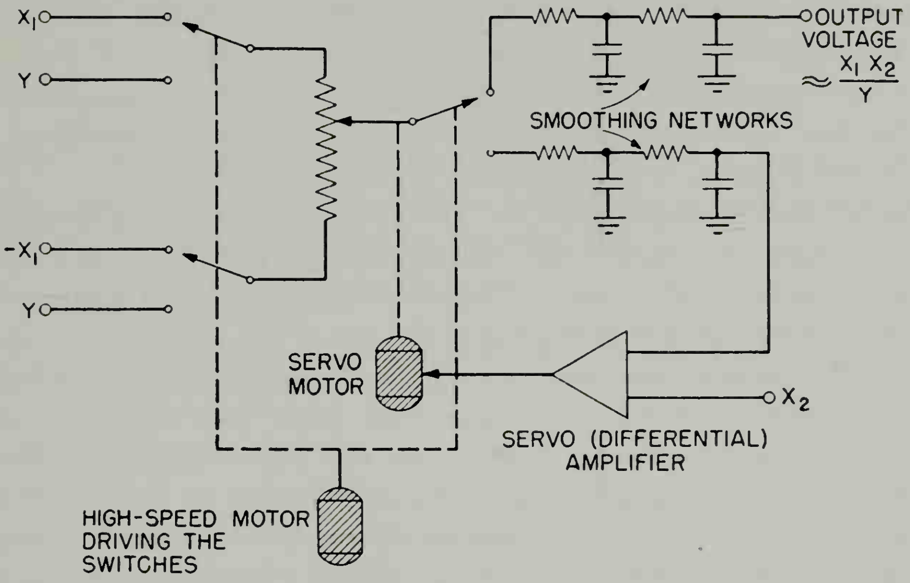
X_1
Y
$-X_1$
Y
OUTPUT VOLTAGE
$\approx \frac{X_1 X_2}{Y}$
SMOOTHING NETWORKS
SERVO MOTOR
SERVO (DIFFERENTIAL) AMPLIFIER
X_2
HIGH-SPEED MOTOR DRIVING THE SWITCHES

FIG. 6.14. Commutated-potentiometer servo. The motor-driven switches may be replaced by vibrating relays. The load resistances in the multiplier and follow-up branches must be equal for good accuracy.

Frequency-sharing Schemes. With voltage-sensitive attenuators whose gain is independent of frequency over a sufficiently wide range, it is again possible to use a single attenuator in a feedback multiplier arrangement. Figure 6.15 shows a possible arrangement. The input machine variable X_1 and the reference voltage Y are simply added and applied to the voltage-sensitive attenuator, which may be a simple variable-gain amplifier or a modulator-demodulator arrangement (see Sec. 6.1). The reference

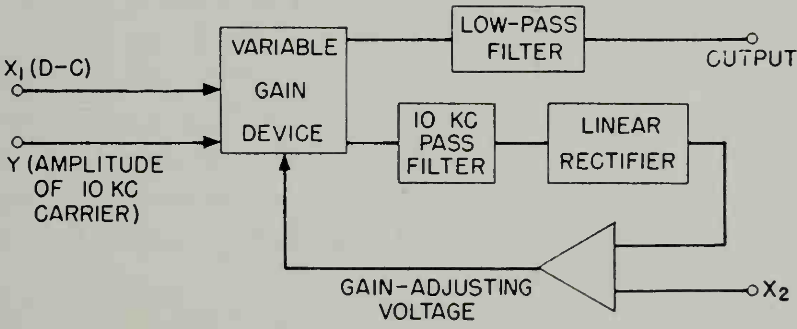

FIG. 6.15. Frequency-sharing feedback multiplier.

voltage Y is the amplitude of an a-c voltage whose frequency is above the desired range of signal frequencies to be used with the multiplier. The resulting high-frequency component of the attenuator output voltage is filtered by a band-pass filter, rectified, and used for adjusting the attenuator gain. The product output voltage, on the other hand, is freed of the high-frequency components by means of a low-pass filter. It is seen that the attenuator transfer function can be made accurately proportional to X_2/Y in this manner; arrangements of this type have permitted accuracies better than 0.2 per cent of full scale.[1]

Review of Multiplying Devices. The feedback multiplying devices most frequently used in "slow" d-c analog computers are various types of multiplying servomechanisms. They are convenient and quite accurate but severely limit the frequency range for one of the variables to be multiplied. The most accurate all-electronic multipliers are the pulsed-attenuator and modulator-feedback types, but these may involve complex circuits using many vacuum tubes. On the other hand, practically all low-cost electronic multipliers now available seem to rely on the nonlinear characteristics of various circuit elements and thus lack precision. There is a definite need for the development of simple and reasonably accurate all-electronic multipliers of low cost for use in computers operating on fast time scales.

6.3. Electronic Division. *Introduction.* Computing elements capable of division must accept two input voltages X and Z, say, and produce an output voltage

$$U = (\text{constant}) \frac{X}{Z}$$

Such a quotient U will increase in absolute value as the absolute value of the divisor Z is decreased. Hence, in view of the given limit on the magnitudes of the computer voltages *no division element will be able to compute X/Z for absolute values of the divisor Z smaller than some minimum value* $|Z|_{\min}$. This proposition must hold no matter what type of computing element is used for division. It is seen that, in general, the quotient X/Z cannot be computed correctly while the divisor Z crosses zero or changes sign during the computation.

Division by Means of Servomechanisms and Similar Feedback Devices. It was seen in Sec. 6.2 that the feedback devices shown in Figs. 6.9 to 6.15 are all adaptable in principle to division as well as to multiplication. As an example of such an arrangement, Fig. 6.16 shows how a servomultiplier may be set up for division. It is seen that there are two alternative input terminals for the dividend. Division schemes of this type are not recommended for general use, since the changes in the feedback loop gain necessarily involved in such arrangements may compromise the design of the feedback circuits.

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[1] Chance, Williams, Hughes, Sayre, and MacNichol, *op. cit.*, Sec. 19.5.

Division by Implicit Computation. The quotient $U = X/Z$ of two machine variables may be obtained from the machine solution of equations like

$$ZU - X\ (\text{constant}) = 0 \tag{6.17}$$

which may be solved continuously by the computer. In arrangements of this type, multipliers are used as such, and servomultipliers may, then, have a constant loop gain for optimum operation.

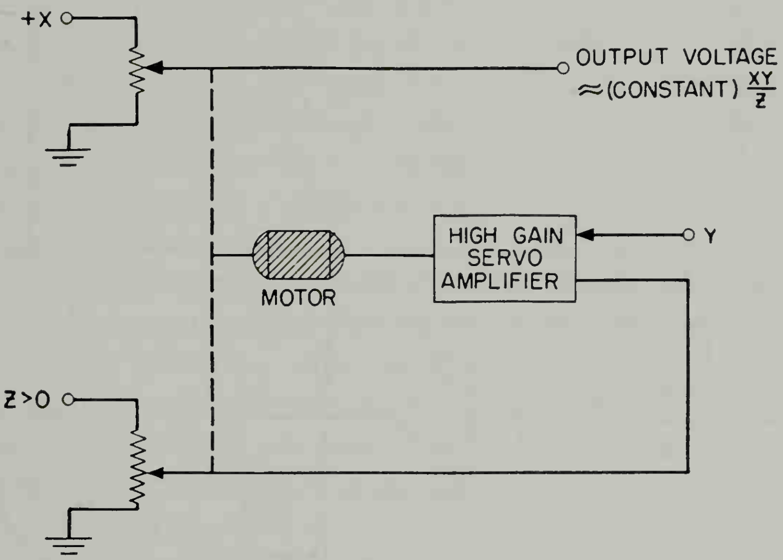

FIG. 6.16. Arrangement for division by means of a high-gain servo loop.

In the circuit of Fig. 6.17, the output voltage of the multiplying potentiometer will be $-U(1 - Z)$, since one end of the potentiometer is grounded, and the center position still corresponds to $Z = 0$. The circuit establishes the relation

$$\begin{aligned} U &= -[X - U(1 - Z)] \\ &= -X + U - ZU \end{aligned} \tag{6.18}$$

so that

$$U = -\frac{X}{Z} \tag{6.19}$$

The circuit is clearly regenerative and can be stable only if the total gain $2 \times \frac{1}{2}(1 - Z)$ around the feedback loop shown is less than 1. It follows that one must have

$$0 < Z \leq 1 \tag{6.20}$$

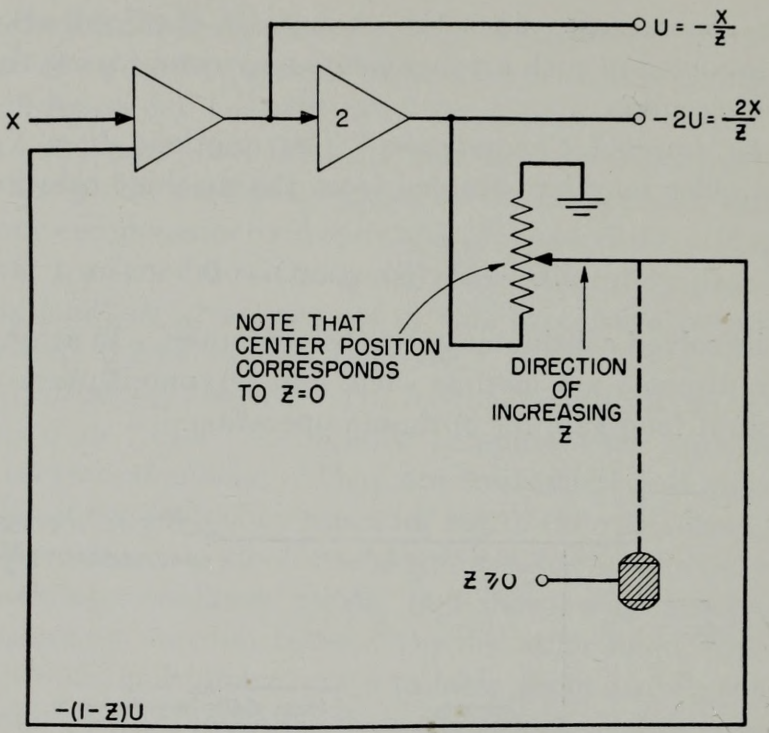
U = -X/Z
X
2
-2U = 2X/Z
NOTE THAT CENTER POSITION CORRESPONDS TO Z=0
DIRECTION OF INCREASING Z
Z ≷ 0
-(I-Z)U

Fig. 6.17. Block diagram of a computer setup for division by implicit computation.

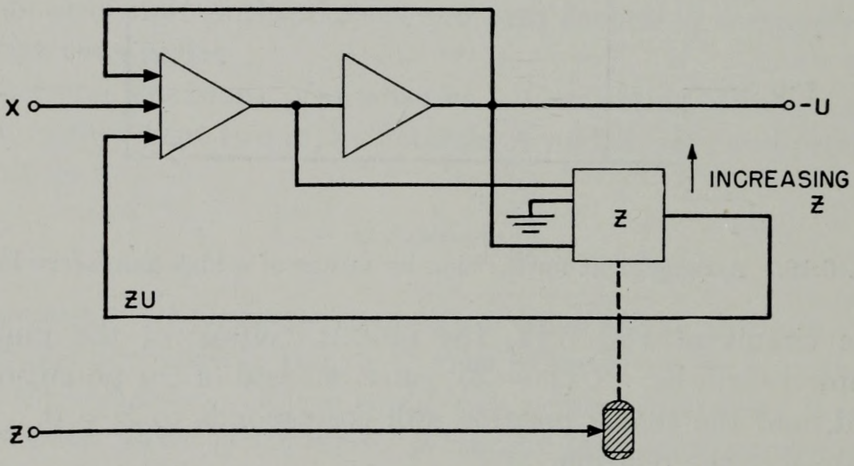

Fig. 6.18. Block diagram of another computer setup for division by implicit computation.

if the machine is actually to solve Eq. (6.18); only one-half the potentiometer winding will be used.

Several variations of this circuit may be used. The arrangement shown in Fig. 6.18 is perhaps more straightforward and establishes the relation

$$-U = X - U + ZU \tag{6.21}$$

so that again

$$U = -\frac{X}{Z}$$

The loop gain is again $(1 - Z)$, so that one must have

$$0 < Z \leq 1 \tag{6.22}$$

as a necessary condition for stability. Again only one-half of the potentiometer winding is used.

If division by a machine variable $Z < 0$ is desired, it is only necessary to reverse the potentiometer terminals in either Fig. 6.17 or Fig. 6.18. In this case, the output voltage will be, in both instances,

$$U = -\left(\frac{X}{-Z}\right) = \frac{X}{Z} \qquad (0 > Z \geq -1) \tag{6.23}$$

Division circuits of this type need two amplifiers (or one push-pull amplifier). These circuits are necessarily regenerative and may develop high-frequency instability for small values of Z. The highest value of Z for which the instability will occur depends on the phase-shift characteristics of the amplifiers used. Such values must not be permitted to occur in the computation.

Division by Reciprocal Multiplication. The division X/Z *of a machine variable* X *by a machine variable* Z *is equivalent to a multiplication* $X(1/Z)$ *by the new function* $1/Z$ *of* Z. Accordingly, all the techniques discussed in Secs. 6.5 and 6.6 for multiplication by functions of variables are applicable to the operation of division as a special case. Assuming that the machine variable Z varies sufficiently slowly with the machine time τ to make servo multiplication possible, it is relatively easy to construct function potentiometers for the function $1/Z$. The design of such potentiometers, together with the necessary precautions against potentiometer loading effects, are discussed in Secs. 6.5 and 6.6.

In spite of the need for a nonlinear potentiometer, this method of division, if it can be applied, possesses important advantages; *viz.*,

1. Devices used for multiplication can be utilized without any circuit changes whatsoever. No complex setups or regenerative circuits are required.
2. If servomechanisms are used, the servo shaft displacement is proportional to Z, so that one or more multiplications by Z or functions of Z can be performed together with the division by Z.

6.4. Practical Computer Servomechanisms for D-c Analog Computers. *Introduction.* The computer servomechanisms used in d-c analog computers are intended to position shafts in accordance with d-c input voltages.

The feedback principle underlying the operation of computer servomechanisms has been described in Sec. 6.2; the methods used for the detailed analysis resemble those used to analyze feedback amplifiers in Sec. 5.4.

Generally speaking, the accuracy with which a servomechanism can position its output shaft as a function of the input voltage will improve with increasing loop gain. Mechanical and electrical inertia of the servomechanism components prevent the servo output from following rapidly changing input signals without a phase lag. With high loop gains such phase lags may result in uncontrolled oscillations (hunting, instability). The phase lag may be counteracted to some extent by speed feedback from an induction generator or tachometer on the output shaft or by special "lead networks" whose operation is similar to that of the lead networks described in Sec. 5.4 in connection with feedback amplifiers. Other equalizing networks may be added to provide higher loop gains at low frequencies. Such "integral" control will decrease the steady-state errors of the servomechanism.

The detailed theory as well as the design of small servomechanisms with specified requirements regarding stability, accuracy, and frequency response has been described very adequately in Vol. 21 of the MIT Radiation Laboratory Series.[1] This section is not intended to duplicate this material but will merely present a number of practical circuits suitable for d-c analog-computer applications.

Basic Circuit Arrangement. Figure 6.19 shows the block diagram of the arrangement most frequently employed in d-c analog computers. The input d-c voltage and the signal feedback from the follow-up potentiometer are mixed to produce an "error signal" at the output of the mixing networks shown. Some integral control[2] is usually introduced at this point in order to decrease the steady-state error of the servomechanism. The d-c error signal so obtained is used to modulate a 60- or 400-cps a-c voltage by means of a synchronous vibrator or chopper of the type described in Sec. 5.6.

The a-c error signal produced in this manner is mixed with the a-c speed feedback signal from an induction generator[3] at the input of a simple a-c amplifier. The amplifier power output stage drives the control field of a two-phase induction motor.

Many variations of this basic circuit are possible. For example, a d-c tachometer, such as a small d-c motor with permanent-magnet field, may replace the induction generator; electrical differentiation of the position feedback signal is rarely used together with wire-wound follow-up potentiometers, since such an arrangement tends to amplify the potentiometer ripple.

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[1] Greenwood, Holdam, and MacRae, *op. cit.*

[2] See *ibid.* for a detailed discussion of integral control and other equalizing techniques.

[3] Refer to *ibid.* An induction generator may be considered as a transformer whose output voltage is proportional to the product of input voltage and rotor speed.

Input Circuits. Figure 6.20 shows a number of different input circuits which accomplish the purposes of mixing the input signal with the various feedback voltages, of introducing integral control, and of modulation. Both single-ended and push-pull circuits are used. The output voltage of the induction generator may be added to the error signal by either a series or a parallel arrangement. Many combinations of such circuits are possible.

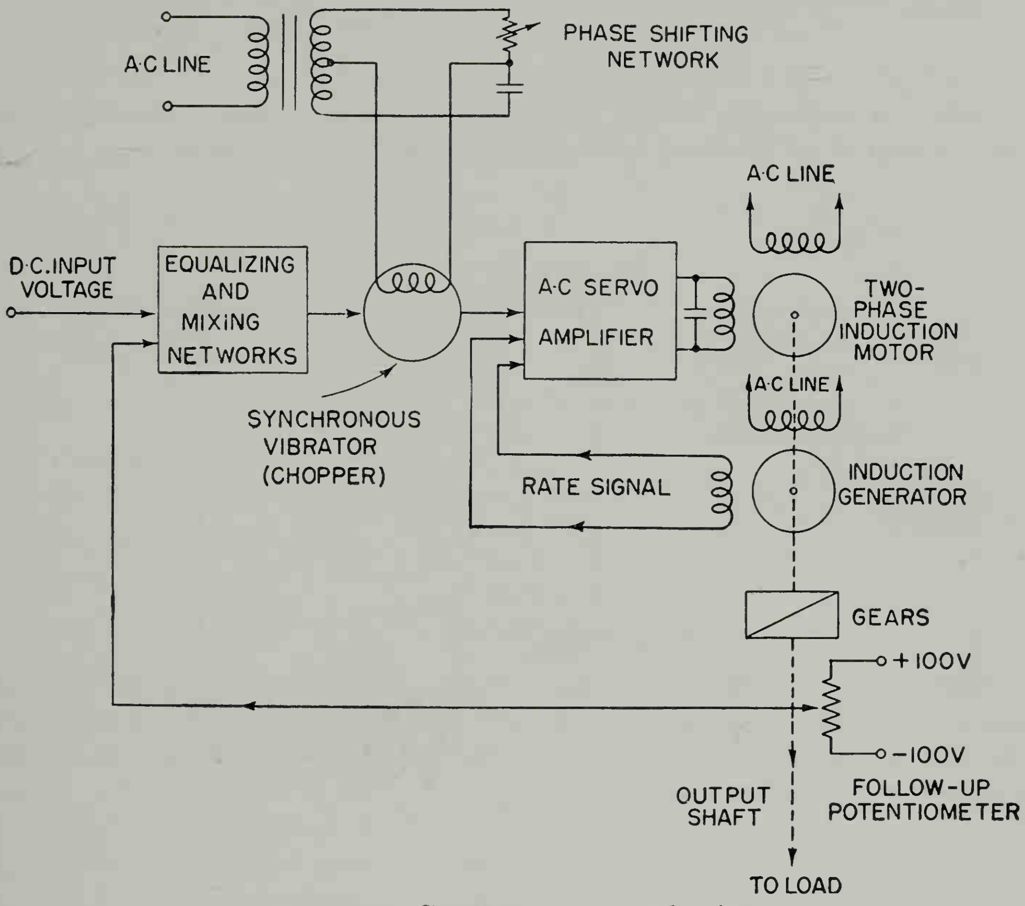

FIG. 6.19. Computer servomechanism.

In the input circuits of Fig. 6.20 feedback from a follow-up potentiometer is indicated; the feedback loop may, instead, include computing elements outside the servomechanism proper.

Servo Amplifiers. Practically any type of a-c amplifier having a maximum gain of about 100 db over the required frequency range and sufficient

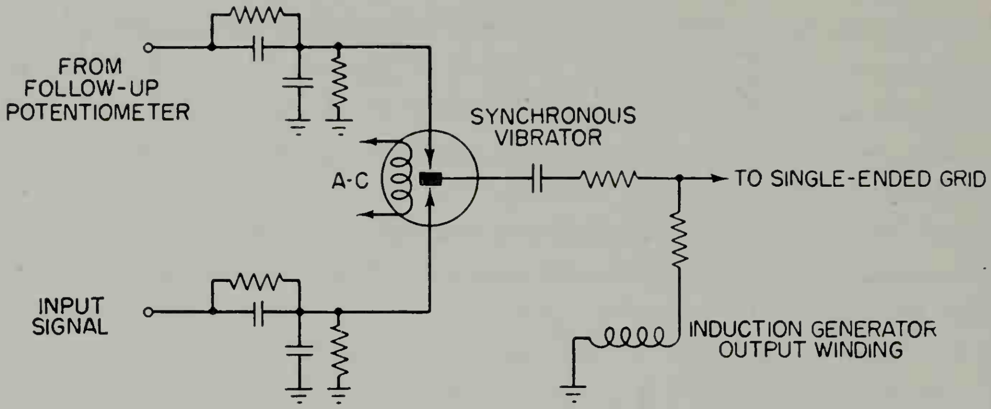

FIG. 6.20*a*. A single-ended input circuit with integral control. The component values will depend on the motor and load characteristics in each case.

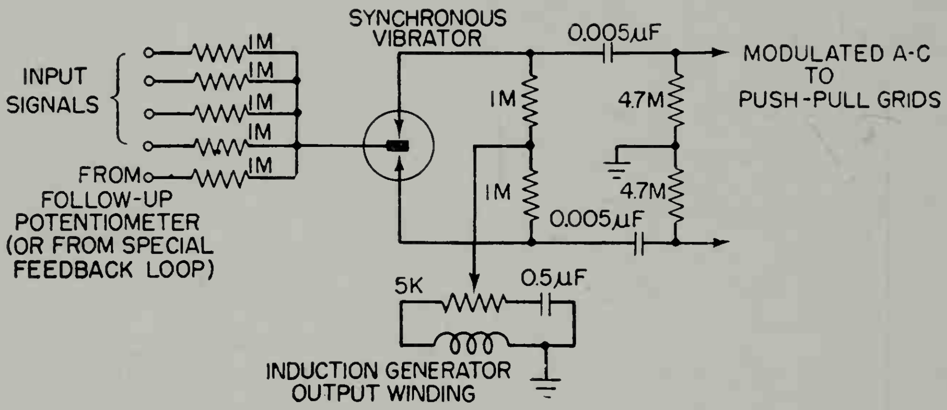

FIG. 6.20*b*. Push-pull input circuit with summing network used by the RAND Corporation (W. F. Gunning and A. S. Mengel, *Internal Report*, RAND Corporation, Santa Monica, Calif., Sept. 8, 1949). Note that the follow-up potentiometer, if used, is loaded by a resistance of 1 megohm.

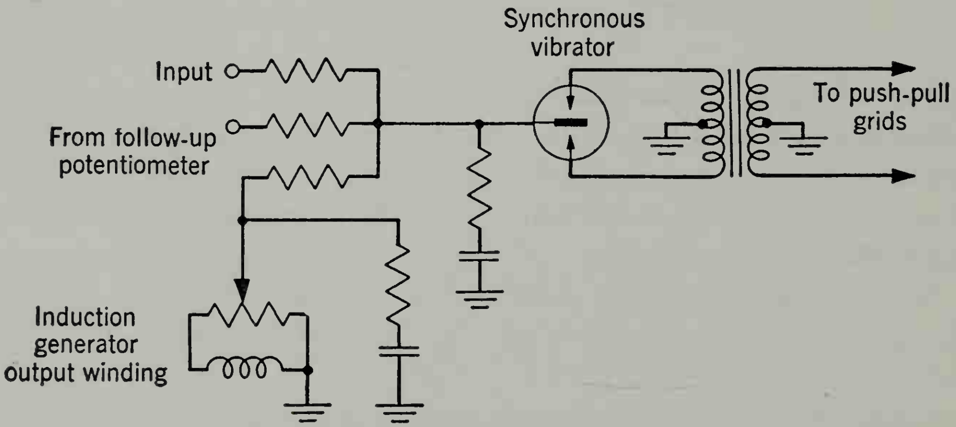

FIG. 6.20*c*. Transformer input circuit.

power output to drive the motor in question can be used as a servo amplifier.

As a relevant example, most public-address amplifiers having a 600-ohm output impedance will serve as excellent low-cost servo amplifiers. It is only necessary that they have the power rating required in each case and that their frequency response be reasonably free of phase shift over a frequency range of about 25 cycles on either side of the carrier frequency chosen (usually either 60 or 400 cps). Such amplifiers have the advantage of self-contained power supplies.

The following simple modifications will be found helpful in converting public-address amplifiers for servo applications:

1. *Limiting resistors* of 20,000 to 150,000 ohms connected in series with at least the output tube grids will help to limit excessively large error voltages.
2. A *simple third-harmonic filter* of the parallel-T type inserted between two voltage amplifier stages (two are needed for push-pull circuits) will reduce noise caused by the third harmonic of the carrier. Such third harmonics are usually generated in the induction generator. Figure 6.21 shows a parallel-T filter for use with a 60-cycle carrier; the equation for the rejection frequency is also indicated and will be found useful for the design of similar filters.

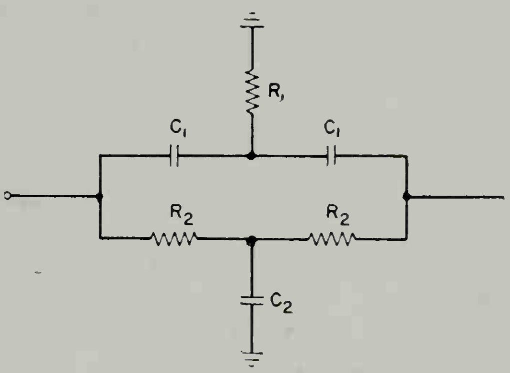

FIG. 6.21. Simple parallel-T third-harmonic suppressor. Useful circuit values for 180-cps rejection are $R_1 = 0.5M$, $R_2 = 1M$, $C_1 = 870$ $\mu\mu$f, $C_2 = 1750$ $\mu\mu$f. The rejection frequency f_r is given by

$$f_r = \frac{1}{\pi R_2} \frac{1}{\sqrt{2C_1C_2}}$$

with $4C_1R_1 = C_2R_2$.

Figure 6.22*a* shows a modified public-address amplifier used as a servo amplifier with excellent results.

Figure 6.23 shows the circuit of a push-pull amplifier specifically designed for servo application[1] in the RAND computer laboratory (see Sec. 8.7).

The use of an input transformer ahead of the servo amplifier permits one to operate the chopper at a lower impedance level; this will tend to reduce the noise introduced in the input circuit. The benefits to be

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[1] Gunning, W. F., and A. S. Mengel, unpublished communication, RAND Corporation, Santa Monica, Calif.

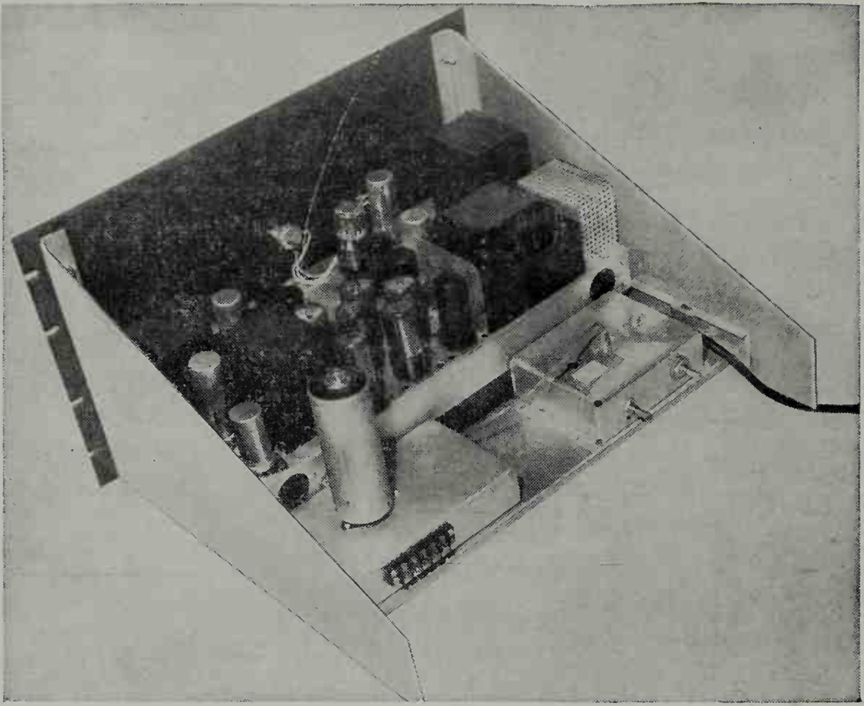

FIG. 6.22*a*. This servo amplifier (Curtiss-Wright Corporation, Airplane Division, Columbus, Ohio) is a modified 45-watt public-address amplifier. The input networks and the synchronous vibrator are arranged on a subchassis in the rear of the amplifier.

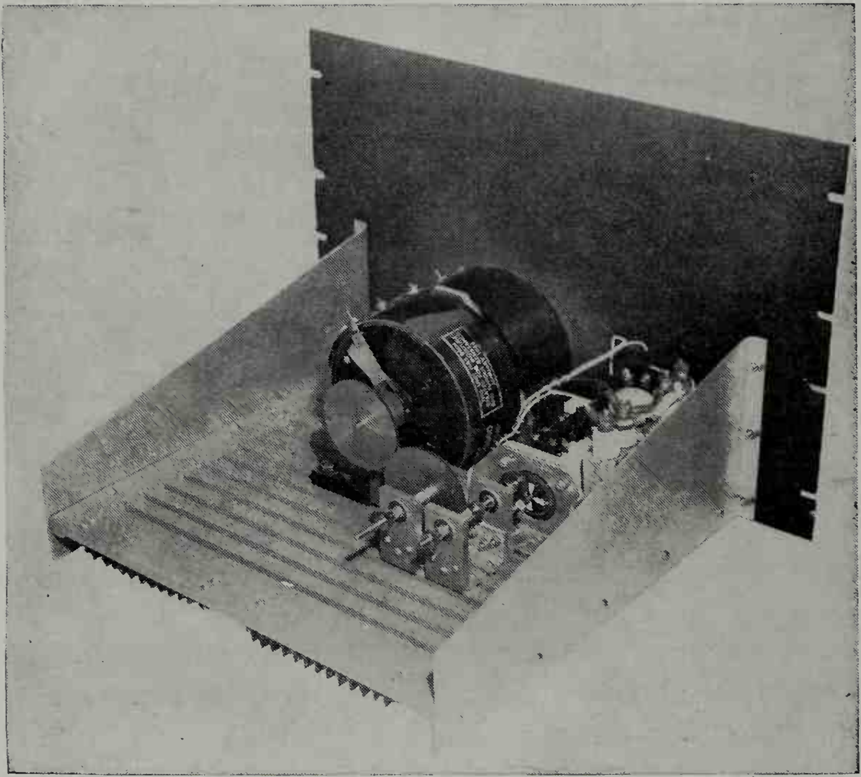

FIG. 6.22*b*. Mechanical components of the same computer servomechanism (Curtiss-Wright Corporation, Airplane Division, Columbus, Ohio) assembled on a slotted base plate for relay rack mounting. Note the gear trains, bearing blocks, and resonating capacitors. A combination motor and induction generator is used.

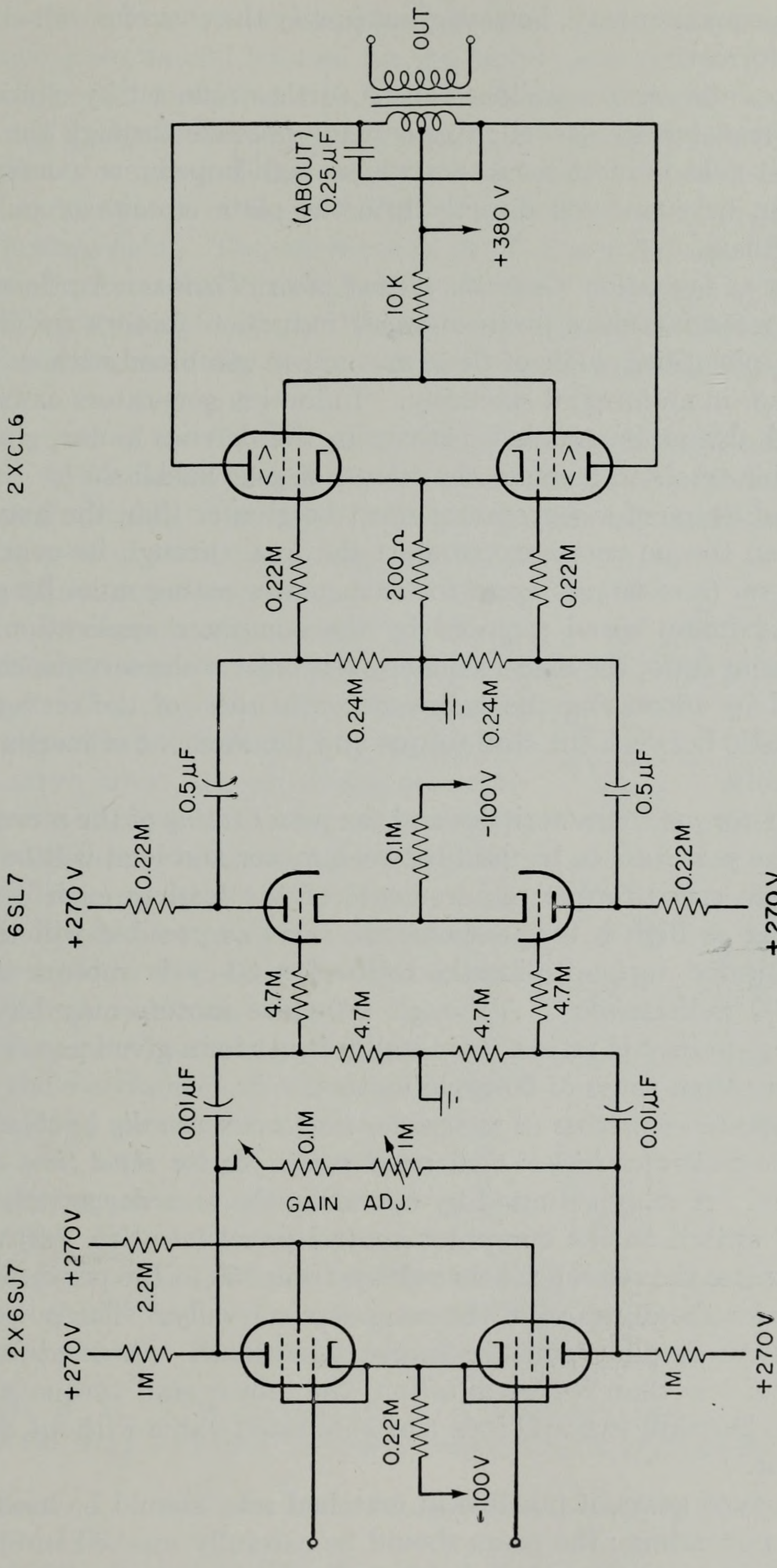

Fig. 6.23. Servo-amplifier circuit from the RAND analog computer (see Sec. 8.7) (Gunning and Mengel, *op. cit.*). This circuit may be used together with the input circuit of Fig. 6.20*b*.

gained in this manner may, however, not justify the cost of a well-shielded input transformer.

The cost of the servo amplifier can be further reduced by eliminating the output transformer as well; this is made possible through the use of modern split-field induction motors whose high-impedance control-field windings can be connected directly into the plate circuits of push-pull power amplifiers.

Servomotors, Induction Generators, and Gear Trains. An increasing number of manufacturers produce small induction motors suitable for computer applications; some of these motors are combined with an induction generator in an integral assembly. Induction generators or tachometers should always be coupled directly to the driving motor, since the differentiation tends to increase the effects of gear backlash.

The *stalled torque* of a servomotor must be greater than the maximum static friction torque necessary to start the load through its gear train. The *maximum* (*zero-torque*) *speed* for each power rating must be greater than the maximum speed required by the computer application. For each given gear ratio, the high-frequency response of the servomechanism is improved by increasing the *torque-to-inertia ratio* of the servomotor; this is the ratio between the stall torque and the moment of inertia of the rotor assembly.

The speed-torque characteristics and the *power rating* of the servomotor as well as the *gear ratio* to be used between motor and load will be determined by the speed-torque requirements of the load in each case. A motor having as high a torque-to-inertia ratio as possible will then be chosen. Suitable torque-to-inertia ratios for 60-cycle motors will be above 20,000 radians/sec^2. Although 400-cycle motors may have still higher torque-to-inertia ratios, their stall torques for a given power rating may be lower than those of 60-cycle motors.

Very substantial increases in servomotor torque can usually be obtained by operating the motors at higher than rated power for the short time of each computer run. A relay actuated by operating the recorder switch or the "compute" switch on the computer control panel (see Sec. 7.2) can be made to increase the reference field voltage from 150 to 190 per cent of the rated value for the duration of the computer run only. The power output of the servo amplifier must be similarly increased. At least for computer runs of less than 3 min duration, the motor stall torque may be increased to between two and four times its rated value without damage to the motor.

Precision-type gears, if possible in matched sets, should be used in all computer applications; the gears should be carefully meshed in order to minimize binding and backlash. Since the inertia of large gears may impair the servo frequency response, gear trains consisting of more than two gears should be used for the higher gear ratios.[1]

Phasing and Equalization. The servomotor control field is usually resonated at the carrier frequency by means of a capacitor in parallel with the primary winding of the output transformer.

The carrier phase in the control field must differ by 90 deg from that in the reference field. The reference fields of all servomotors in a computer are usually connected directly to the a-c line. The carrier phase in the control fields may be adjusted by means of a phase-shifting network, such as the one shown in Fig. 6.19 in the a-c supply used for the synchronous vibrators.

The values of the forward gain, rate, and integral feedback of a servomechanism for stable operation with good frequency response are usually computed roughly; the exact values are then obtained through final adjustments of the equalizing networks. These adjustments may be made with the aid of a servomechanisms tester[2] by checking the servo response to sinusoidal input voltages. If such a device is not available, a d-c step-function voltage may be applied to the servo input; the equalizing networks are then adjusted until the servo output follows the input voltage as closely as possible without excessive overshoot. Some other methods for testing servomechanisms are discussed in Vol. 21 of the MIT Radiation Laboratory Series.[3]

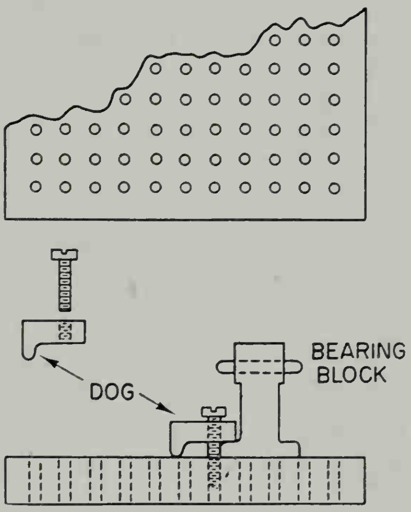

FIG. 6.24. Means of attaching servo components to a drilled and tapped base plate. Note that the components may be turned in any direction on the base plate.

Mechanical Construction. Figure 6.22*b* shows the electromechanical components of a computing servomechanism set up on a *slotted base plate* which allows for a wide variety of component arrangements. As an alternative, Fig. 6.24 shows a method of attaching servo components to *drilled base plates* by means of dogs and screws. In the authors' opinion, such drilled base plates are preferable to slotted plates. The drilled plates not only are cheaper to produce but also allow greater convenience and flexibility in the arrangement of the components. The dogs shown in Fig. 6.24 are easily made by drilling and cutting stock pieces of aluminum having the profile required. The holes in the base plate itself may or may not be tapped, although the former makes for more convenient assembly. Bases and shafts may be lined up by means of a simple T square or similar device.

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[1] James, H. M., N. B. Nichols, and R. S. Phillips, *Theory of Servomechanisms*, pp. 131–133, MIT Radiation Laboratory Series, Vol. 25, McGraw-Hill, New York, 1947.

[2] Korn, G. A., and T. M. Korn, Modern Servomechanism Testers, *Elec. Eng.*, September, 1950.

[3] Greenwood, Holdam, and MacRae, *op. cit.*

Both slotted and drilled base plates as well as suitable bearing blocks, motor mounts, etc., for small servomechanisms are available commercially from a number of manufacturers.

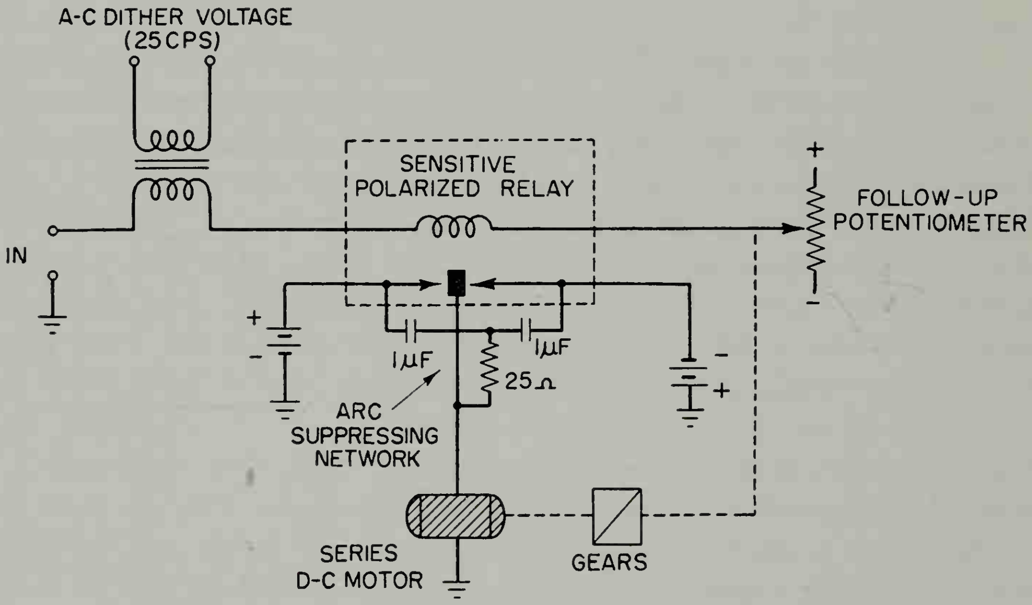

FIG. 6.25*a*. Simple relay servomechanism with dither. The optimum dither frequency and amplitude will depend on the relay spacing, sensitivity, and time delay.

Other Types of Servomechanisms. Relay servomechanisms using a sensitive polarized relay in a circuit like that of Fig. 6.25*a* to control d-c motors may be useful in certain special applications because of their relative simplicity. The frequency response of such devices may be made reasonably flat up to 2 or 3 cps. In order to minimize the effects of static friction in the relay, an a-c voltage (the so-called dither voltage) is introduced into the relay coil so as to vibrate the relay armature continuously. The dither frequency is made higher than the highest frequency to which there is appreciable servo response. If the contacts of the sensitive relay cannot handle the motor current, the sensitive relay can control a power relay which in turn controls the motor. Although relay circuits may be less complex than vacuum-tube circuits, these devices are generally not so reliable as vacuum-tube amplifiers, and their sensitivity and frequency response may not be satisfactory except in special cases.

If an a-c power source is available, a-c motors are usually preferred to d-c motors in order to eliminate the maintenance problems usually associated with brushes and commutators. The a-c amplifiers used for controlling two-phase motors require a high-voltage d-c supply which may or may not be self-contained in each servo amplifier. If servomotors are operated at high power levels, the d-c supplies needed for amplifiers using vacuum tubes may become bulky and expensive. In such cases, recourse may be had to thyratron control of a-c motors,[1] but careful shielding and isolation of the thyratrons will be necessary in order to prevent their interaction with other computing elements. The use of *saturable reactors* for direct control of the alternating currents in the motor fields may be preferable. Figure 6.25*b* shows how the alternating current in a motor field may be effectively controlled by a d-c input voltage. Although some internal noise is inherent in saturable reactor modulators, its results can be effectively reduced by including a d-c preamplifier in the servo loop. Ordinary operational amplifiers may conceivably be adapted for this purpose. The use of saturable reactors will eliminate the need for synchronous vibrators and the attendant maintenance problems as well as the requirement for high-power d-c supplies. A d-c tachometer may be used for speed feedback.

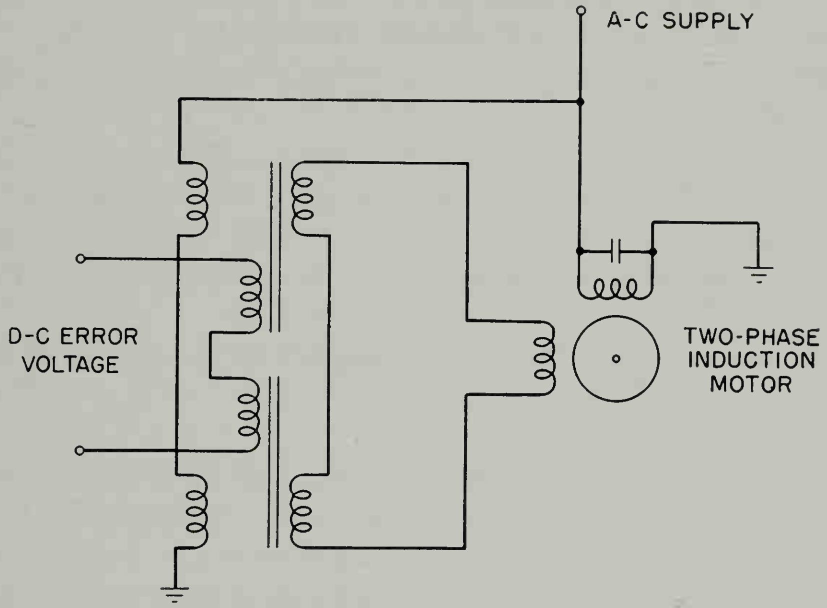

FIG. 6.25*b*. Use of a saturable transformer to control the a-c voltage applied to a motor field in accordance with a d-c input voltage.

Finally, serious consideration should be given to the use of commutated potentiometer servomechanisms of the type described in Sec. 6.2.

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[1] *Ibid.*, Chap. 12.

Servo-driven Potentiometers. The potentiometers used with computer servomechanisms must satisfy the requirements of mechanical ruggedness, freedom from microphonism, accuracy, resolution, and stability already discussed in Sec. 4.1 in connection with constant coefficient setting potentiometers. In addition, servo-driven potentiometers must satisfy special requirements directly related to their utilization in the servomechanism, *viz.*,

1. The static and dynamic torques needed to position the potentiometer wiper must be as low as consistent with reliable operation. A static torque of 2 oz-in. per potentiometer may be considered as a reasonable value.
2. The potentiometer windings must sustain the wiper friction for a reasonable life period without changing their properties. At the rotational speeds involved in the normal operation of most d-c analog computers, a useful life of at least 300,000 revolutions may be expected.
3. A minimum of noise should be generated by the motion of the wiper over the potentiometer winding.

The construction of potentiometers satisfying these requirements has made considerable progress in recent years and is described by F. E. Dole in Vol. 17 of the MIT Radiation Laboratory Series.[1]

The use of potentiometers to generate special functions is discussed in Secs. 6.5 and 6.6.

If helical-type potentiometers are to be driven by servomechanisms, special precautions must be taken to prevent damage to the internal stops in the potentiometers caused by accidentally driving the wiper assembly into these stops. The stall torque of a servomotor, when applied to the potentiometer through a gear train, can be very appreciable and will almost certainly destroy the potentiometer. Special systems of mechanical stops and/or limit switches have been devised for this purpose. Probably the easiest solution is to use single-turn potentiometers. Such potentiometers may be obtained with accuracies and resolutions better than 0.1 per cent of full scale for resistance values between 10,000 and 100,000 ohms; they can be fitted with "bridges" which permit continuous rotation. It may be advisable to fit each servomechanism with a neon overload bulb similar to those used with d-c amplifiers (see Sec. 6.2) to indicate when each potentiometer wiper is about to leave the winding; error indicators of the type described in Sec. 5.6 may be even more useful.

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[1] Blackburn, J. F., *Components Handbook*, Chap. 8, MIT Radiation Laboratory Series, Vol. 17, McGraw-Hill, New York, 1949.

Potentiometer Loading. Errors in computations due to loading of servo-driven potentiometers cannot in general be corrected through changes in calibration as in the case of constant coefficient setting potentiometers (see Sec. 4.1). The effects of potentiometer loading may be reduced by connecting resistors in series with either the source or the load[1] through the use of special feedback circuits[2] and by a number of other circuit arrangements.[3] In the case of servomultipliers, *probably the best solution to this problem is to make sure that the follow-up and multiplying potentiometers are all loaded by equal resistances;* such an arrangement will automatically eliminate the loading errors in the output voltage. (This does not apply to function generators; see Sec. 6.5.) In addition, *the use of center-tapped follow-up and multiplying potentiometers is strongly recommended.*

Note that, although equal loading of follow-up and multiplying potentiometers will eliminate errors in *multiplication* due to potentiometer loading, *the servo shaft displacement will not be accurately proportional to the servo input voltage if the follow-up potentiometer is loaded.* If the servo shaft displacement is read on a linearly calibrated dial, a loading correction (see Sec. 4.1) must be applied to the dial reading to obtain the servo input voltage.

Accuracy Improvements. The accuracy of servo multiplication can sometimes be improved by using tapped potentiometer function generators (see Sec. 6.6) as multiplying potentiometers; the effective taper of such potentiometers can be adjusted to correspond closely to that of the follow-up potentiometer.

Again, a *correction term* to compensate for differences in taper and/or loading between the multiplying and the follow-up potentiometer may be measured and then generated by a special function generator on the servo shaft in the manner illustrated in Fig. 6.32 (Sec. 6.5).

The use of commutated potentiometer servomechanisms should also be considered in this connection.

6.5. Basic Types of Function Generators. *Introduction.* An imposing number of the relations to be set up on d-c analog computers involve the generation of a voltage as a function of time or as a function of another voltage. In other cases, the multiplication of a voltage by such a function is required.

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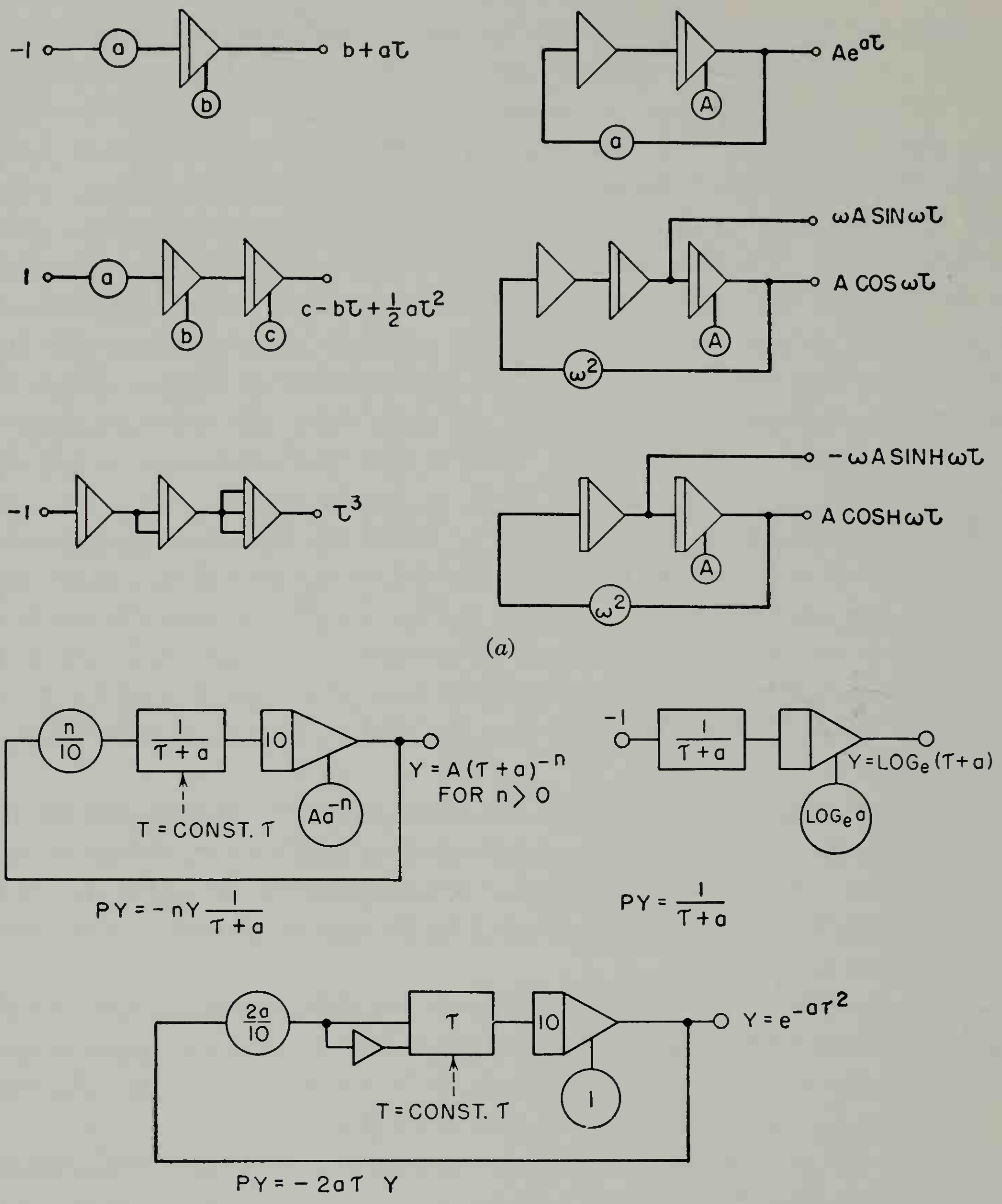
[1] Dole, F. E., and L. A. Nettleton, Reducing Potentiometer Loading Error, *Rev. Sci. Instruments*, **18**: 332, 1947.

[2] Follin, J., Internal Memorandum, Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Md.

[3] Dole and Nettleton, *op. cit.*



(*a*)

(*b*)

FIG. 6.26. Examples of functions of the time τ generated as solutions of simple differential equations. The function potentiometers indicated in the first two circuits of (*b*) may be replaced by division loops (see Sec. 6.3) so that only linear potentiometers are needed.

The computing elements capable of performing these operations are called *function generators.*

Function generators are characterized by their accuracy, flexibility, reliability, cost, and frequency response. The remainder of this chapter presents several available methods of function generation and discussions of their principal advantages and disadvantages.

The first class of function generators described in the following are *all-electrical devices.* Their good high-frequency response makes them suitable for computer applications requiring fast time scales, such as repetitive computers, but, generally speaking, their accuracy leaves much to be desired.

The second class of function generators are *electromechanical devices* which are capable of considerable accuracy. Their frequency response, on the other hand, is limited by the inertia of mechanical components.

Implicit Function Generation. Many functions of the time can be generated in a d-c analog computer as solutions of simple differential equations. Figures 6.26*a* and *b* show a number of block diagrams of computer setups suitable for generating frequently used functions of the time. Many other functions can be generated in this manner.

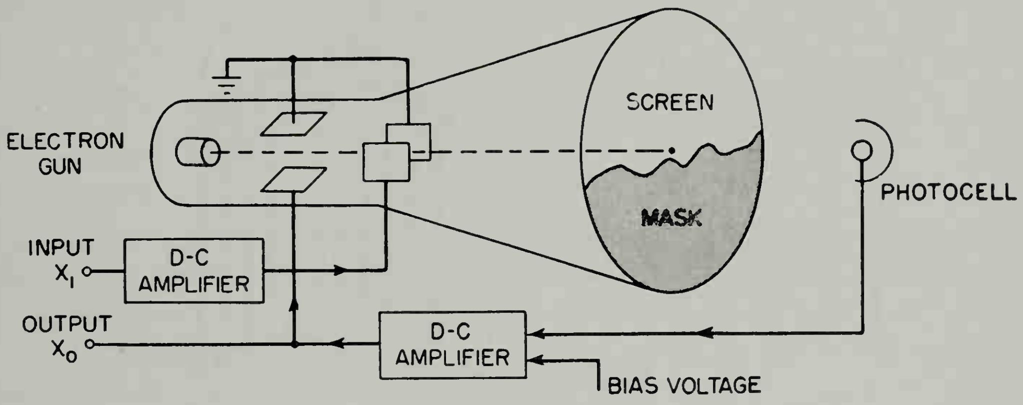


FIG. 6.27*a*. Block diagram of a basic photoformer.

If a function generator for the inverse $g(Y)$ of a given function $f(X)$ is available, the latter can usually be obtained by an implicit solution of the machine equation

$$g[f(X)] - X = 0 \tag{6.24}$$

The division schemes described in Sec. 6.3 are examples of this method.

Nonlinear Circuit Elements. The voltage-current characteristics of certain nonlinear circuit elements such as tubes, rectifiers, and thyrite resistors can be used in many different circuit arrangements to establish functional relationships between input and output voltages.

The circuit characteristics of only a few nonlinear circuit elements can be reproduced with an accuracy better than 1 per cent. Perhaps the most useful devices of this kind are based on the use of vacuum or crystal diodes as *switching elements* rather than as continuously variable nonlinear resistance elements. Function generators based on this principle are discussed in detail in Sec. 6.7.

Cathode-ray Function Generators: Photoformers. Figure 6.27*a* shows the block diagram of the basic cathode-ray function generator or *photoformer.*[1] The input voltage X_1 is applied between the horizontal deflection plates of a cathode-ray tube through a suitable d-c amplifier. The voltage X_o between the vertical deflection plates is taken to be the output voltage; this voltage is made to vary as a function

$$X_o = f(X_1)$$

of the input voltage X_1 by a feedback arrangement which forces the electron beam to follow the boundary of an opaque mask placed over the lower portion of the cathode-ray screen.

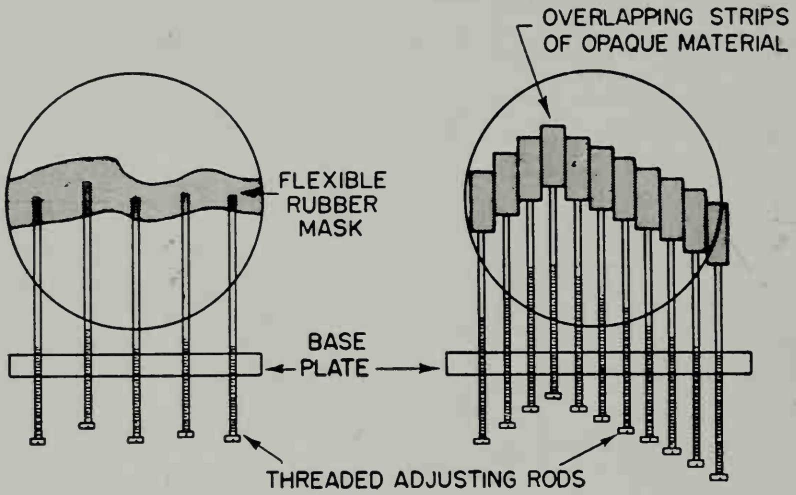


Fig. 6.27*b*. Adjustable mask for photoformers.

The bias voltage indicated in Fig. 6.27*a* tends to force the electron beam upward away from the mask. As soon as the spot on the cathode-ray tube screen emerges from behind the mask, however, a photocell mounted in front of the screen applies an error voltage across the input terminals of the vertical deflection d-c amplifier. This error voltage is phased so that the beam is forced downward toward the edge of the mask. *The feedback loop comprising the cathode-ray tube, the photocell, and the vertical deflection amplifier will, then, tend to keep the spot on the screen traveling just below the edge of the mask as the input voltage X_1 (horizontal deflection voltage) changes.*

Since the deflections of the electron beam produced by the deflection voltages X_1 and X_o are approximately proportional to these respective voltages, *the output voltage X_o is forced to vary as a function*

$$X_o = f(X_1)$$

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[1] Mynall, D. J., Electronic Analog Computing, *Elec. Eng.*, August, 1947; Sunstein, D. E., Photoelectric Waveform Generator, *Electronics*, **22**: 100, 1949.

of the input voltage X_1; the function $f(X_1)$ is determined by the shape of the mask boundary. If the equation of the latter in rectangular Cartesian coordinates ξ and η with origin in the center of the cathode-ray screen is

$$\eta = F(\xi)$$

then

$$f(X_1) = \frac{1}{k_\eta} F(k_\xi X_1) \tag{6.25}$$

where k_ξ and k_η are the horizontal and vertical deflection sensitivities

$$k_\xi = \frac{\partial \xi}{\partial X_1} \qquad k_\eta = \frac{\partial \eta}{\partial X_o} \tag{6.26}$$

respectively. It is, then, possible to obtain a mask for the generation of the function $f(X_o)$ by cutting the corresponding curve

$$\eta = F(\xi) = k_\eta f\left(\frac{1}{k_\xi}\,\xi\right) \tag{6.27}$$

out of a piece of opaque cardboard or plastic.

Functions may be generated in this manner with an accuracy of the order of 1 per cent for input voltages containing frequency components in excess of 100,000 cps if suitable amplifiers are used.

Better accuracy might be possible if Eq. (6.27) is not relied on to yield the shape of the mask for the desired function. *A flexible or otherwise adjustable mask could be adjusted until the measured value of the output voltage X_o is equal to the correct value $f(X_1)$ for each value of the input voltage X_1.* Errors due to changes in the deflection sensitivities k_ξ and k_η with the applied voltages as well as errors due to the finite gain of the feedback loop could be eliminated in this manner. Figure 6.27*b* shows two possible types of adjustable masks.

The actual circuit of a photoformer (essentially as developed at the Massachusetts Institute of Technology[1]) is shown in Fig. 6.28. The use of a multiplier photocell in connection with a very simple push-pull deflection amplifier results in ample feedback gain. The P-11 cathode-ray tube screen used could be replaced by a P-5 screen for high-frequency applications.[2] The maximum error (distortion) observed with this circuit was about 2.5 per cent (with a fixed mask).

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[1] Cambridge, Mass.

[2] MacNee, A. B., An Electronic Differential Analyzer, *Proc. IRE*, **37**: 1315, 1949. MacNee has also recently developed a variable photoformer mask consisting of 90 metal shims clamped into a suitable frame.

The all-electronic photoformers are, for all practical purposes, free from frequency-response limitations. Although they permit the generation of one voltage as a function of another, no multiplication is possible. Photoformers are very simple devices, and their reliability and accuracy are being improved.

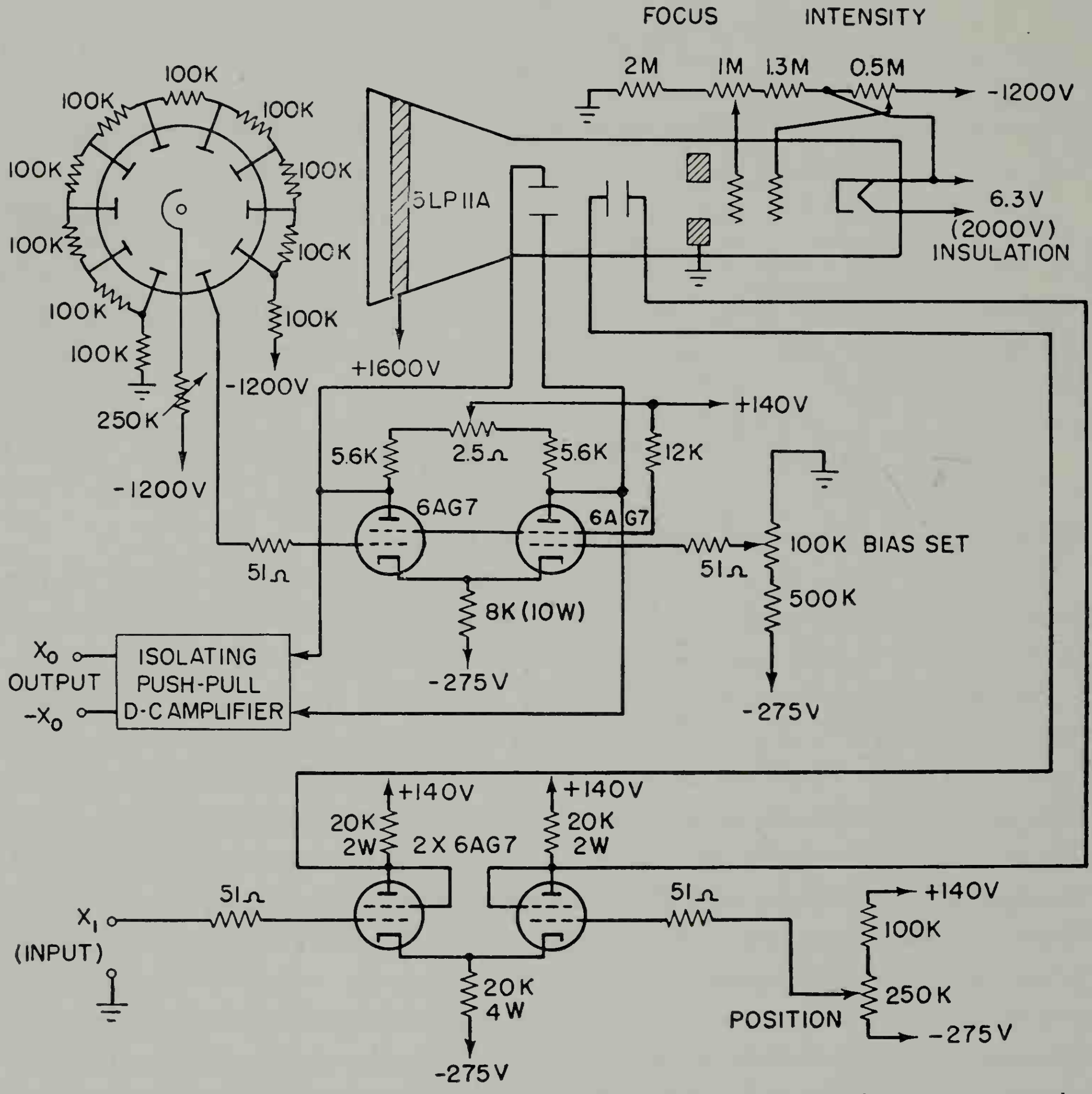

FIG. 6.28. Practical photoformer circuit. Both positive and negative output may be obtained. (Based on the circuit used in the MIT repetitive computer described in Sec. 8.5.)

Other Cathode-ray Function Generators. The *monoformer* developed by the Philco Corporation[1] operates essentially like the photoformer, but the external mask and photocell are replaced by a mask and a collector electrode, respectively, inside the cathode-ray tube envelope. Since the monoformer mask is not easily replaceable, a monoformer is capable of generating a single function only; the accuracy is comparable to that of a photoformer.

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[1] Philadelphia, Pa.

The Raytheon Manufacturing Company[1] has developed another type of cathode-ray function generator in which a flat electron beam is narrowed down by a mask in accordance with a given function as the beam is deflected by an input voltage. The output voltage is obtained as a function of the beam current. The accuracy is of the order of 1 per cent.

Special cathode-ray tubes like the monoformers and the Raytheon tube may be useful as function generators in special-purpose analog computers, such as relatively accurate square-law modulators for multiplying circuits (see Sec. 6.1), and possibly in quarter-square and logarithmic multipliers (see Sec. 6.1) if good frequency response is more important than accuracy.

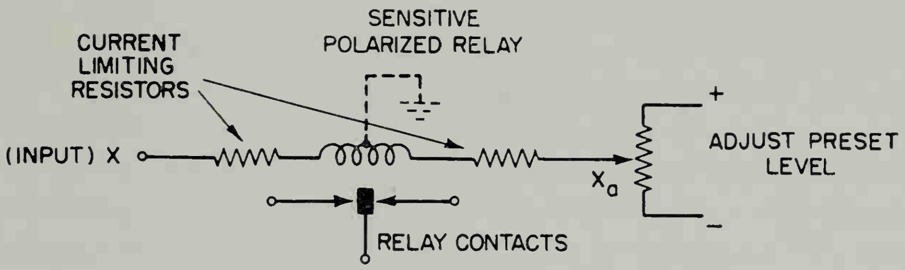

FIG. 6.29. Basic relay computing circuit. For best results, the relay should be inserted in the output circuit of a suitable differential amplifier.

Relays as Function Generators. Relays can be used to introduce step voltages or to change interconnections between computing elements whenever certain of the machine variables (voltages) or differences of machine variables reach predetermined levels. Figure 6.29 shows a circuit arrangement for actuating a sensitive polarized relay when the difference between the input voltage X and the preset voltage X_a exceeds zero by a few hundredths of a volt. For better results, the relay should be inserted into the output circuit of a symmetrical differential amplifier (see Sec. 5.3).

Relays can be used only in computations performed on a time scale so slow as to render the relay contact closing and opening time delays effectively negligible. In computers operating on faster time scales, the relays can be replaced by suitable *electronic switches,* such as the limiting diodes described in Sec. 6.7.

Tapered Function Potentiometers. Potentiometers or voltage dividers can be constructed so that the fraction of the total resistance tapped off by the potentiometer arm varies as a nonlinear function of the shaft displacement. Such potentiometers can be used to multiply an input voltage applied across the potentiometer resistance element by a function of the shaft displacement.

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[1] Waltham, Mass.

Function potentiometers as well as various other electromechanical function generators are usually driven by computer servomechanisms (see Sec. 6.4) whose shaft displacements are proportional to the respective input voltages. Functions of the machine time τ can also be generated by driving the corresponding function potentiometers with a synchronous motor connected either to the 60-cycle a-c line or to the power output stage of a special stable timing oscillator of the tuning-fork type. The latter arrangement is almost mandatory for good accuracy, since the 60-cycle line frequency is capable of instantaneous variations as large as 0.7 per cent.

Wire-wound elements for function potentiometers are given the proper "taper" (or rate of resistance change with respect to shaft displacement) by changing the shape of the card, the pitch of the winding, or even the wire diameter in accordance with the desired function. The construction of tapered function potentiometers is discussed in detail in Vol. 17 of the MIT Radiation Laboratory Series;[1] tapered wire-wound function potentiometers for a number of frequently used functions are commercially available.

Simple tapered function potentiometers can generate only functions which vary monotonically between two values determined by the voltages at the two input terminals of the potentiometer winding. Nonmonotonic functions can, however, be generated by employing one or more *taps* in the potentiometer winding; the output voltage into a load of infinite impedance will, then, vary monotonically between each pair of voltages at the taps.

Function Potentiometers Based on Linear Resistance Elements with Controlled Wiper Displacement. A linear potentiometer winding may be used for function generation if the displacement of the potentiometer wiper is made to vary as the desired function of an input shaft rotation. A cam or a linkage can be used to transmit the input shaft motion in the desired manner. A stylus linked to a potentiometer through a linkage, a rack-and-pinion drive, or a worm-gear arrangement may even be used for manual following of any reasonable curve plotted on a moving chart. The potentiometer can then generate a voltage as a function of the chart displacement. The usefulness of this *input table* is not restricted to the generation of a single function; it may be regarded as a type of "universal function generator."

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[1] Blackburn, J. F., *Components Handbook*, Chap. 8 by F. E. Dole, MIT Radiation Laboratory Series, Vol. 17, McGraw-Hill, New York, 1949.

Automatic Curve Followers; Cylindrical-type Function Potentiometers. The necessity for operating an input table manually can be eliminated by using a special servomechanism (photoelectric curve follower) for keeping the stylus on the plotted curve. A much simpler type of automatic curve follower is illustrated by the automatic input table used, *e.g.*, in the RAND analog computer, and shown schematically in Fig. 6.30.

This device utilizes a drum rotated about its axis by a servomechanism whose input voltage is the argument of the function to be generated. A linear resistance element constituting the potentiometer winding is mounted parallel to the drum axis in the manner shown in Fig. 6.30. A wire is cemented over a graph of the curve to be tracked, and the graph paper is placed on the drum with the independent variable axis in the direction of drum rotation. As the drum rotates in correspondence to the independent variable, the linear resistance element is always parallel to the drum axis directly over the correct value of the dependent variable on the graph paper.

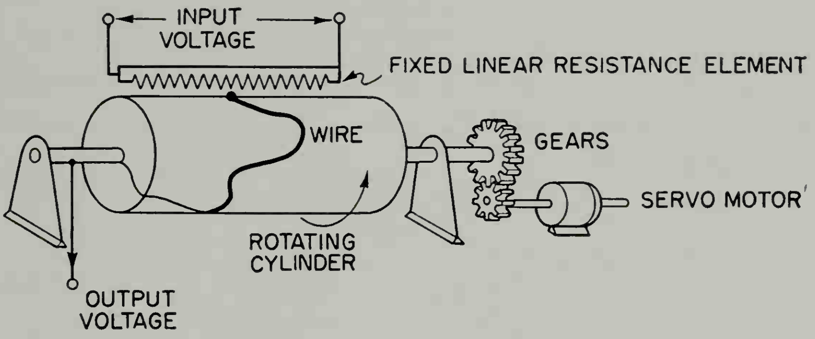

Fig. 6.30. Reeves-type function generating potentiometer.

The wire on the curve then acts as the wiper of a linear potentiometer[1] tapping off the proper fraction of an input voltage applied across the linear resistance element as the latter is pressed against the drum and wire by light spring pressure. As the drum rotates, this wiper moves across the potentiometer in a manner determined by the given curve.

The RAND input table utilizes the servo-positioned drum of a Reeves servo recorder or output table (see Sec. 7.3). With a linear resistance element of 20,000 ohms resistance and linear to within 0.1 per cent, it is possible to multiply a voltage by any reasonable function of another voltage with an accuracy of about 0.15 per cent of full scale.

The capability of this and similar devices ("universal function generators") of generating any desired reasonable function makes them particularly useful in flexible multipurpose computers.

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[1] Gunning and Mengel, *op. cit.*

The preparation of the wire for any given function is simple. A length of 0.015-in.-diameter nichrome wire is first straightened by stretching it beyond the yield point. The wire is then fitted to the curve drawn on a piece of graph paper lying flat on a table. The wire is held to the paper by means of small split pins and is covered with a layer of cement (one part of General Cement No. 30-8 Radio Service Cement to one part of No. 31-8 Solvent) which secures it to the paper even after the pins are removed. When the cement has dried, the top surface of the wire is laid bare through the use of abrasive cloth. The graph paper is then ready to be placed on the drum.

The Reeves Instrument Corporation,[1] which first invented such cylindrical-type potentiometers, manufactures small function potentiometers based on the principles just described, for any reasonable function to the customer's specifications, as well as the larger automatic input tables. A servo-driven lathe is used to cut a groove for the wire into a small cylinder made of insulating material.[2]

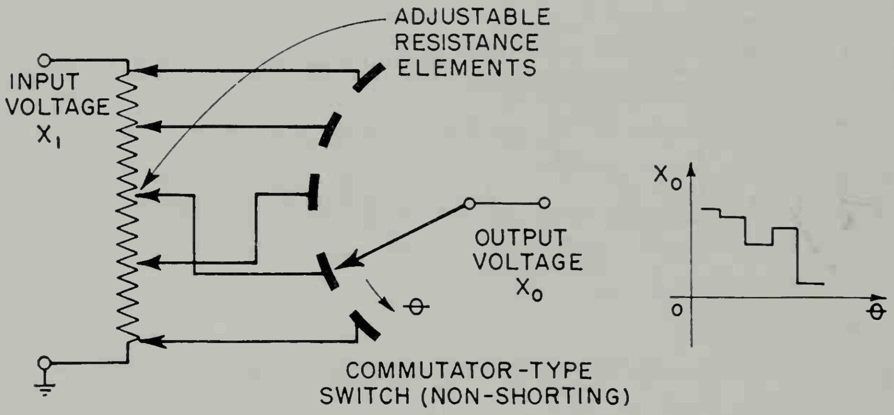

FIG. 6.31*a*. Simple switch-type universal function generator.

Switch-type Function Generators. If it is sufficient to approximate a given function by a step function (the step function may, for instance, be smoothed out by subsequent integration), a function potentiometer may be replaced by a simple tap switch rotated in accordance with the argument of the given function (see Fig. 6.31*a*). The tap voltages are adjusted in accordance with the values of the given function for the corresponding shaft displacements. Simple universal function generators may be constructed in this manner.

For better accuracy, a linear "interpolating potentiometer" may be connected between successive taps by means of two switch arms in the manner shown in Fig. 6.31*b*. The interpolating potentiometer is geared up from the switch shaft so that it makes a complete revolution during the time required by each switch arm for moving between corresponding positions on successive taps or commutator bars. The output voltage is taken from the wiper of the potentiometer, which serves to interpolate between the values of the output voltage obtained at the tap points.

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[1] New York, N.Y.

[2] Frost, S., A Compact Analog Computer, *Electronics*, July, 1948.

A method for utilizing *tapped potentiometers* as universal function generators by a somewhat similar interpolation scheme is discussed in detail in Sec. 6.6.

Discussion of Function Potentiometers. Effects of Potentiometer Loading. The discussion given in Sec. 6.3 of the properties desirable in multiplying potentiometers also applies to potentiometers used for function generation, with the exception of the linearity requirement. Potentiometer function generators are reliable and accurate, but their application is often limited by the frequency response of the electromechanical system or servomechanism used for positioning the potentiometer shaft.

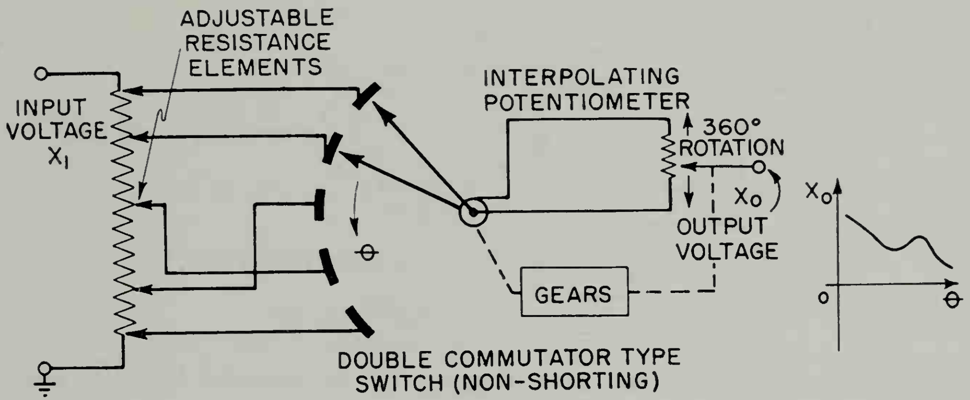
ADJUSTABLE RESISTANCE ELEMENTS
INPUT VOLTAGE X_1
INTERPOLATING POTENTIOMETER
360° ROTATION
X_0
OUTPUT VOLTAGE
θ
GEARS
DOUBLE COMMUTATOR TYPE SWITCH (NON-SHORTING)
X_0
0
θ

FIG. 6.31*b*. Switch-type universal function generator with interpolating potentiometer.

The effects of potentiometer loading on the accuracy of servo-driven potentiometer function generators cannot be eliminated simply by loading the follow-up and function potentiometers with equal resistances as in the case of servomultipliers. A loading correction may have to be applied to the function to be generated. Again, the servo shaft displacement will no longer be accurately proportional to the servo input voltage if the follow-up potentiometer of the servomechanism is loaded. *These effects must be carefully taken into consideration in connection with the use of input tables of the type shown in Fig.* 6.30. The relation between the input voltage and the servo shaft displacement may in this case be computed in the manner shown in Sec. 4.1, and a corresponding correction may be applied to the given function. Universal function generators of the tap-switch and tapped-potentiometer types can usually be adjusted to take care of loading corrections automatically whenever necessary.

Useful function generators can be constructed by using a function potentiometer or tap switch to vary the feedback and/or input resistors of an operational amplifier. Such function generators do not require loading corrections.

A Method for Improving the Accuracy of Function Generators. The accuracy with which a given function may be generated can often be substantially improved by a universal function generator. Figure 6.32 shows how the universal function generator may be used to add a small correction to the output voltage of the original function generator. The correction function must be determined by checking the output voltage against the desired function values for a number of values of the argument. Good accuracy is possible, because errors due to the correction function generator are multiplied by a small number by the constant coefficient potentiometer shown.

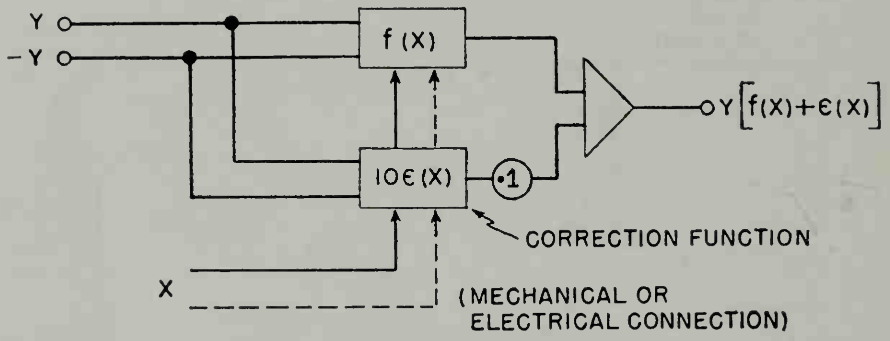

Fig. 6.32. Use of a correction function generator to improve function generator accuracy.

This method has been successfully applied[1] to the reduction of errors due to nonlinearity and loading in servomultipliers and servo-driven potentiometer function generators, and should be applicable to all-electrical function generators as well.

Special Function Generators Based on Linear Potentiometers. Frequently, simple linear potentiometers may be used for generating special functions. The output voltage of a *loaded linear potentiometer* was seen in Sec. 4.1 to be a nonlinear function of the shaft displacement. Figure 6.33*a* illustrates how a number of different functions may be generated by connecting different load resistances to a linear potentiometer and by returning the load to various points in the circuit. Figure 6.33*b* shows how various other useful functions may be generated or approximated by special connections of linear potentiometers.[2]

Circuits of the types just illustrated are, generally speaking, more applicable to computers used for special purposes rather than for flexible, multipurpose machines.

[1] Gunning and Mengel, *op. cit.*

[2] Chance *et al.*, *op. cit.*, Chap. 12.

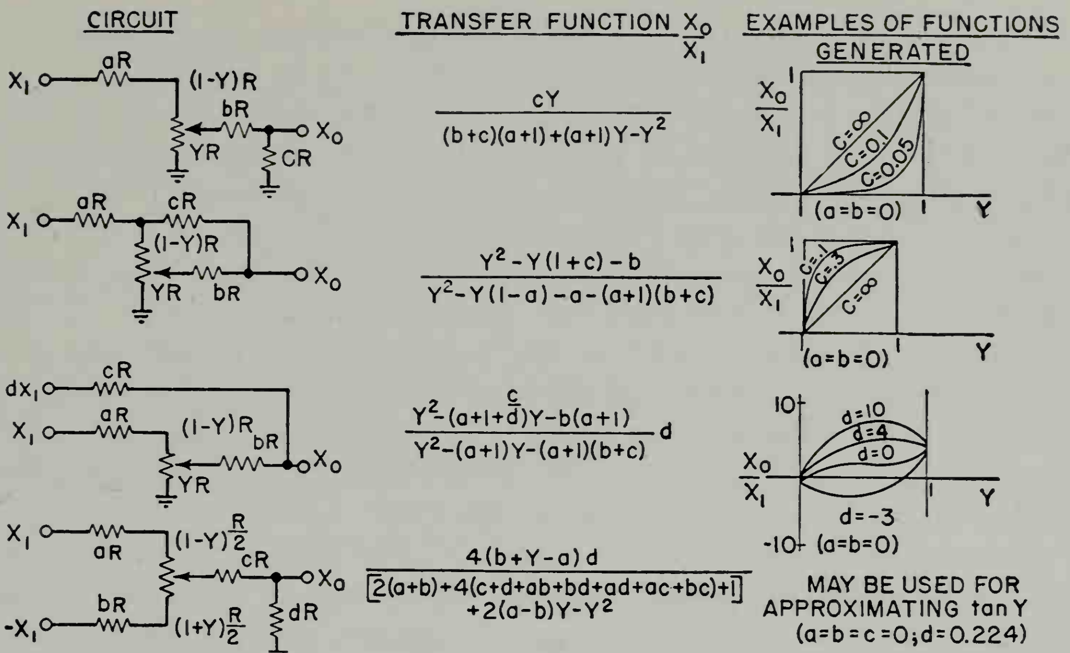
CIRCUIT
TRANSFER FUNCTION Xo/X1
EXAMPLES OF FUNCTIONS GENERATED
X1 aR (1-Y)R bR Xo YR CR
cY / (b+c)(a+1)+(a+1)Y-Y²
Xo/X1 1 C=∞ C=0.1 C=0.05 (a=b=0) 1 Y
X1 aR cR (1-Y)R Xo YR bR
Y²-Y(1+c)-b / Y²-Y(1-a)-a-(a+1)(b+c)
Xo/X1 1 C=1 C=3 C=∞ (a=b=0) 1 Y
dX1 cR X1 aR (1-Y)R bR Xo YR
[Y²-(a+1+c/d)Y-b(a+1)] / [Y²-(a+1)Y-(a+1)(b+c)] d
10 Xo/X1 d=10 d=4 d=0 1 Y d=-3 -10 (a=b=0)
X1 aR (1-Y)R/2 cR Xo -X1 bR (1+Y)R/2 dR
4(b+Y-a)d / [2(a+b)+4(c+d+ab+bd+ad+ac+bc)+1] +2(a-b)Y-Y²
MAY BE USED FOR APPROXIMATING tan Y (a=b=c=0;d=0.224)

FIG. 6.33*a*. Use of loaded linear function potentiometers for approximating various nonlinear functions. The potentiometer setting is equal to the machine variable Y (Greenwood, Holdam, and MacRae, *op. cit.*).

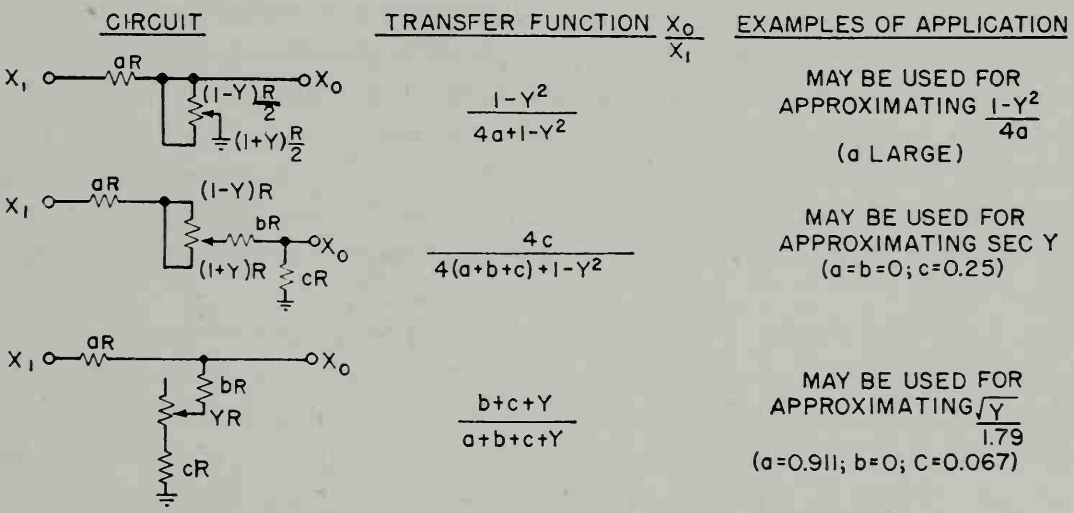
CIRCUIT
TRANSFER FUNCTION Xo/X1
EXAMPLES OF APPLICATION
X1 aR (1-Y)R/2 (1+Y)R/2 Xo
1-Y² / 4a+1-Y²
MAY BE USED FOR APPROXIMATING (1-Y²)/4a (a LARGE)
X1 aR (1-Y)R bR Xo (1+Y)R cR
4c / 4(a+b+c)+1-Y²
MAY BE USED FOR APPROXIMATING SEC Y (a=b=0; c=0.25)
X1 aR Xo bR YR cR
b+c+Y / a+b+c+Y
MAY BE USED FOR APPROXIMATING √Y/1.79 (a=0.911; b=0; C=0.067)

FIG. 6.33*b*. Miscellaneous circuit arrangements permitting the use of simple linear potentiometers for generating nonlinear functions (Chance *et al*, *op. cit.;* Greenwood, Holdam, and MacRae, *op. cit.*). The networks are adjusted in each case to match the desired functions. The potentiometer setting is equal to the machine variable Y.

Special Devices for Generating Sinusoidal Voltages. The generation of sinusoidal voltages of the time or of other machine variables is particularly important in connection with the solution of problems involving Fourier analysis or synthesis, problems involving frequency-response tests of automatic control systems, and problems involving coordinate transformations (see Sec. 6.8).

The possibility of generating sinusoidal functions of the machine time by implicit computation has been mentioned; the case of special sine and cosine function potentiometers is discussed in Sec. 6.8. Sinusoidal functions of the time may also be generated by means of *electronic oscillators* of the phase-shift type,[1] although the design of oscillators stable with respect to amplitude and frequency presents somewhat of a problem, especially if the frequencies required are below 0.5 cps. The correct phasing of electronic oscillators may raise still another problem.

Rotatable transformers, such as synchros and resolvers, will modulate an a-c carrier voltage with sinusoidal functions of the shaft displacement. Together with suitable phase-sensitive demodulators or detectors, such devices may be used as d-c function generators; they are somewhat less subject to wear than function potentiometers.

The input a-c carrier may be modulated with another d-c voltage, so that multiplication is possible; several applications of this type are discussed in Sec. 6.8. D-c offset voltages originating in the modulators and demodulators present a serious difficulty. The d-c frequency response is limited, as with potentiometers.

Figure 6.34 shows the block diagram of a flexible and convenient low-frequency oscillator using synchros to generate sinusoidal d-c voltages of the time for testing the frequency response of simulated or actual automatic control mechanisms.[2] A rate servo with adjustable tachometer feedback is used to rotate a synchro transmitter at any desired rate between 0.001 and 10 cps. The speed of rotation can be calibrated conveniently by stroboscopic reference to the 60-cps line frequency through the use of a simple neon bulb and a small stroboscopic disk on the high-speed shaft.

A 5,000-cps carrier voltage is used. The three-phase output of the synchro transmitter is impressed on the stator winding of a second synchro. The rotor winding of this second synchro is held stationary and affords a particularly convenient means for adjusting the phase of the d-c output voltage X_1 obtained through the use of a phase-sensitive detector circuit.[3] A calibrated potentiometer attenuator permits one to adjust the amplitude of the output voltage X_1.

A second sinusoidal d-c output voltage X_2 is obtained by demodulating the stator voltage of the first synchro directly. This voltage is useful as a *reference voltage. This voltage may, for instance, be used as an input voltage*

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[1] See, for instance, F. E. Terman, *Radio Engineers' Handbook,* McGraw-Hill, New York, 1943.

[2] Korn and Korn, *op. cit.*

[3] Refer to Chance *et al., op. cit.,* Chap. 14, for a discussion of phase-sensitive detectors.

*for a real or simulated automatic control system whose output phase and amplitude can then be measured by comparison with the calibrated phase and amplitude, respectively, of the voltage X_2.**

Functions of Several Machine Variables. A voltage Y, say, which is equal to a given function of not one but several machine variables $X_1, X_2, \ldots$ can usually be obtained or at least approximated by means of judicious combinations of summing and/or multiplying circuits with devices generating functions of a single variable. Thus the function

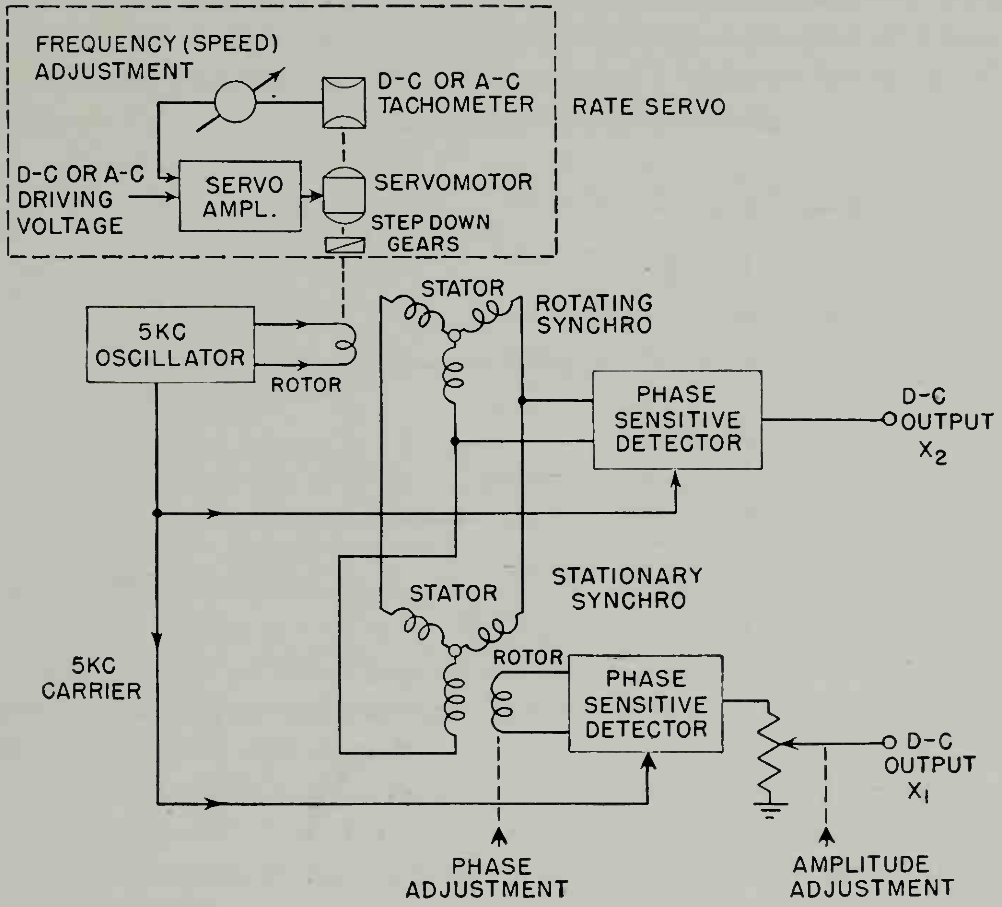

FIG. 6.34. Block diagram of a low-frequency oscillator used for testing automatic control circuits.

$$Y = \sin(X_1 + X_2)$$

is obtained by first adding the voltages X_1 and X_2 and then generating the sine of their sum.

Functions of several variables given by empirical data (tables or curves) can also frequently be approximated by combinations of sums and products of functions of a single variable. As an alternative, a function

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* Korn and Korn, *op. cit.*

$Y(X_1,X_2)$ of two machine variables X_1 and X_2 may be generated as a function of the shaft displacement X_1 for several fixed values of X_2 by a number of function potentiometers. A tapped potentiometer, whose shaft displacement is proportional to X_2, can then be used for *interpolating* approximate values of $Y(X_1,X_2)$ as shown in Fig. 6.35. Instead of using separate function potentiometers for different values of X_2, it is often possible to use a single cylindrical-type function potentiometer (Fig. 6.30) with separate contact wires corresponding to each function $Y(X_1,X_2 = \text{constant})$.[1] The tapped function potentiometer shown in Fig. 6.35 may also be replaced by one of the commutating switch arrangements shown in Fig. 6.31.

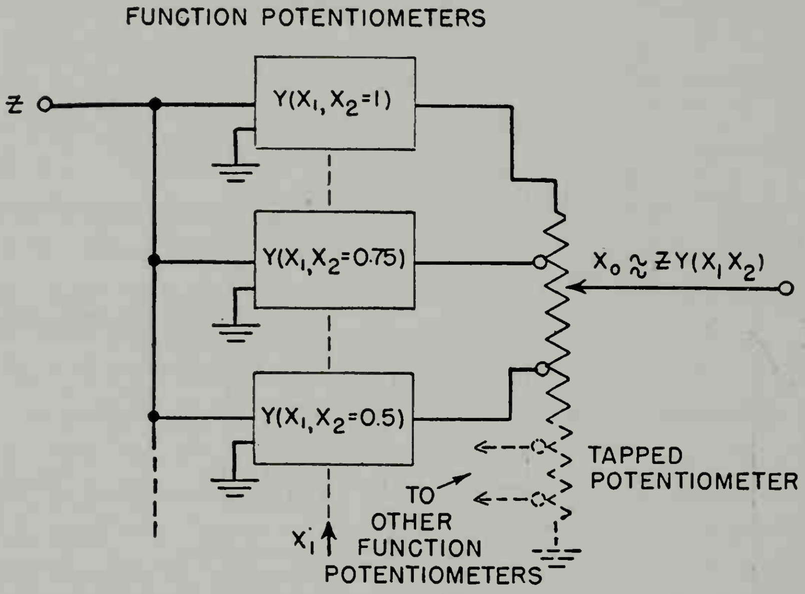

Fig. 6.35. Circuit for multiplying a voltage Z by a function $Y(X_1,X_2)$ of two machine variables.

The Variable-density-type Cathode-ray Function Generator. It is also possible to generate a voltage Z proportional to a function $f(X,Y)$ of two input voltages applied to the horizontal and vertical deflection plates of a cathode-ray tube. A translucent mask, whose density varies with position in a manner determined by the desired function, is placed between the screen of the cathode-ray tube and a photocell. The photocell output voltage will then vary in the desired manner as the voltages X and Y are varied.[2] Although the accuracy of such a function generator is not too good (1 to 10 per cent), the frequency response is excellent (up to at least 100,000 cps), and the device is suitable for use in repetitive computers. In particular, functions like (constant)XY,

[1] Gunning and Mengel, *op. cit.*

[2] Wallmann, H., An Electronic Integral-Transform Computer and the Practical Solution of Integral Equations, *J. Franklin Inst.*, **250**: 45, 1950.

(constant)X^2Y, etc., could be generated by a variable-density-type cathode-ray function generator which can thus serve as a simple electronic multiplier. Masks having opacity variations over a range of 200 to 1 can be produced by photographic means. (See also Fig. 6.36.)

Noise Generators. Voltages representing noise in systems simulated by d-c analog-computer setups may be introduced as functions of time by means of function generators set up to simulate typical noises in the systems to be simulated. Various types of noise may also be introduced by means of noise generators such as noisy diodes, photocells, or thyratrons in magnetic fields, with or without output filters.

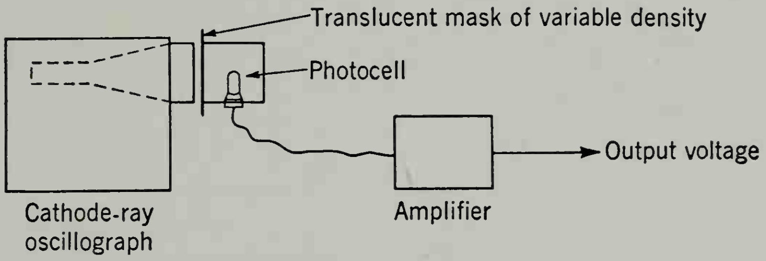

Fig. 6.36. A simple variable-density-type cathode-ray function generator. The two input voltages are the horizontal and vertical deflection voltages of a direct-coupled cathode-ray oscillograph.

6.6. Tapped Potentiometers as Universal Function Generators.[1] *Introduction.* Quite often the delivery times of special tapered function potentiometers are prohibitively long if quick tests involving empirically obtained functions are desirable, and the prices are often out of all proportion to those of associated equipment.

In view of these facts, it would be desirable to have a function generating potentiometer which could be easily made up in the laboratory at reasonable cost and which could be set up for a wide variety of desired functions, including functions obtained from empirical data. The following is a description of a new type of function generating potentiometer which is believed to satisfy these requirements.

An inexpensive linear potentiometer is tapped nineteen to thirty-nine times. A set of adjustment networks permits the voltages at each tap to be adjusted to correspond to the values of the desired function for the respective angular displacements. The potentiometer output voltage will be correctly set up for the desired function values at the taps. Moreover, the potentiometer will interpolate approximately linearly between the function values set up at the taps, and accuracies better than 1 per cent are possible for reasonable gradient changes. The adjustment networks contain small variable potentiometers or rheostats, so that the function to be generated may be changed at will. If no such changes are desired, fixed resistors may be substituted for the variable ones.

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[1] This section is based on the author's paper on the same subject in *Rev. Sci. Instruments,* **21** (No. 1): 77–81, January, 1950, originally *Boeing Document* D-10041. The author is grateful to *The Review of Scientific Instruments* and to Dr. Gaylord P. Harnwell, editor, for permitting this material to be reproduced here and also to E. C. Wells of the Boeing Aircraft Company, Seattle, Wash., for his permission to publish *Boeing Document* D-10041.

The authors are also grateful to P. G. Redgemont of the British Admiralty Signal and Radar Establishment for pointing out to them that tapped function potentiometers were used in England as early as 1933.

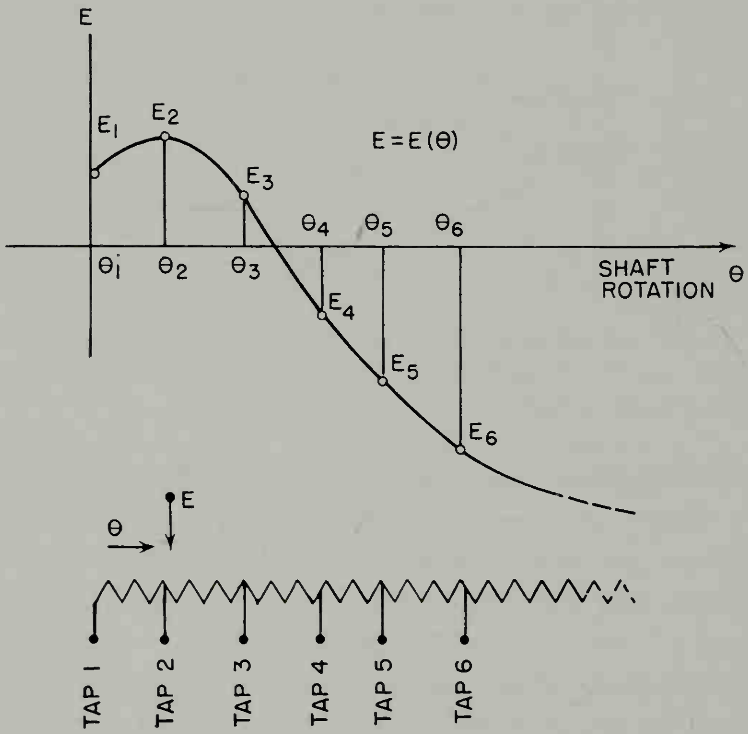

FIG. 6.37. Linear interpolation by means of tapped potentiometers.

The principle of the universal function generator is illustrated in Fig. 6.37, whereas Fig. 6.38 shows the actual circuits of the adjustment networks for a number of typical cases.[1] The "parallel" arrangement of the adjustment potentiometers shown in Fig. 6.38*a* is the most flexible one and permits a wide variety of functions to be set up. Its disadvantages are (1) the low impedance presented to the source by the parallel potentiometers and (2) the danger of burning out the main potentiometer winding if the full source voltage appears across a small section of the winding. Both these disadvantages are overcome by the "series-parallel" arrangement of Fig. 6.38*c* at the expense of flexibility. In this arrangement, the adjustment potentiometers are essentially connected in series across the source, so that each tap voltage can be varied only through a fraction of the source voltage without rewiring. This arrangement is particularly well suited for fixed function potentiometers. The arrangement of Fig. 6.38*b* may present an even higher impedance to the source, but it is not so easily adjusted as that of Fig. 6.38*a*.

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[1] Korn, G. A., The General Difference Analyzer, *Curtiss-Wright Rept.* P 537-V-46, 1948.

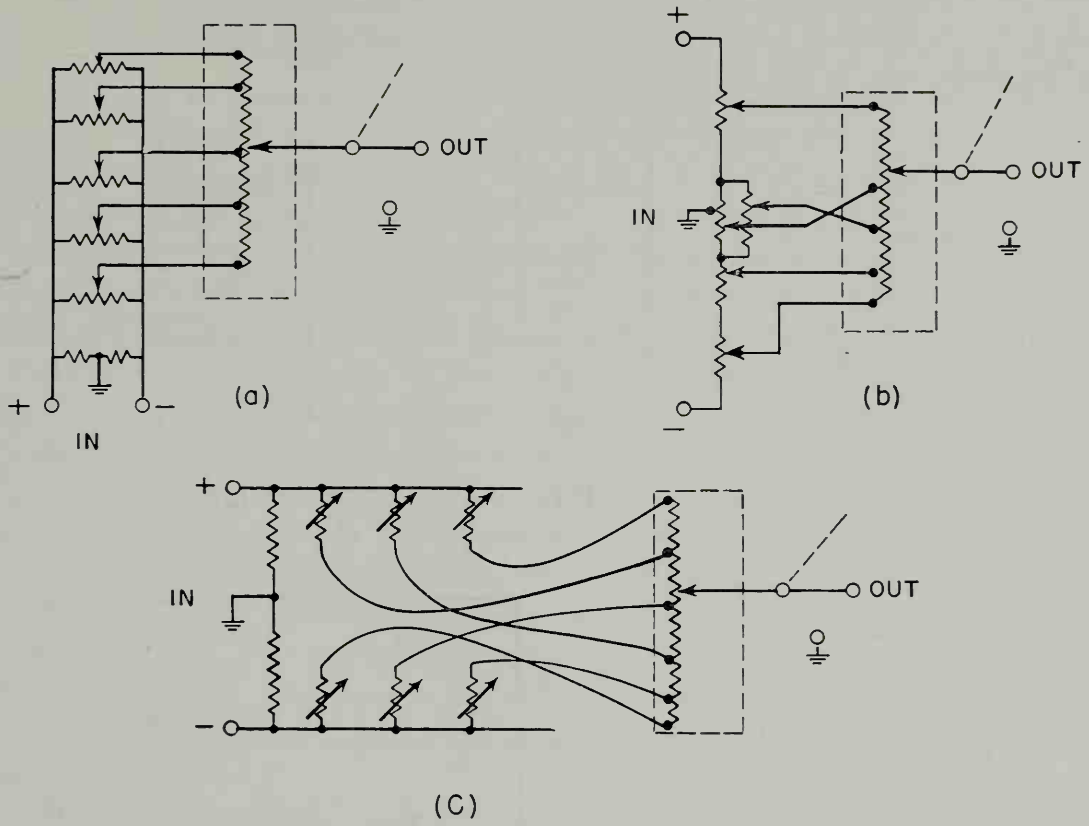

Fig. 6.38. Universal function generator circuit diagrams.

The following advantages of the new function potentiometers should be noted:

1. Relatively low cost potentiometers can be adapted. Five per cent accuracy with respect to the linearity of the card and the location of the taps is ample, since the final accuracy is not seriously affected by this.
2. The effect of potentiometer loading on the taper of potentiometers is largely eliminated, since the tap voltage adjustments may be made with the load connected. This results in considerable savings and design simplifications in many cases.
3. The new function potentiometer possesses a high degree of flexibility. Different characteristics can be set into instruments, computers, etc., by simply changing the tap voltage adjustments. Tables of these adjustments giving desired functions may be compiled for reference.

or complete, interchangeable adjustment networks may be retained for reuse (see page 270).

The choice of resistance values for the tapped potentiometers and adjustment networks, as well as the effects of loading, will be discussed.

Design Considerations. The choice of resistance elements for the tapped potentiometer and its associated networks must satisfy several requirements, which will be taken up in turn.

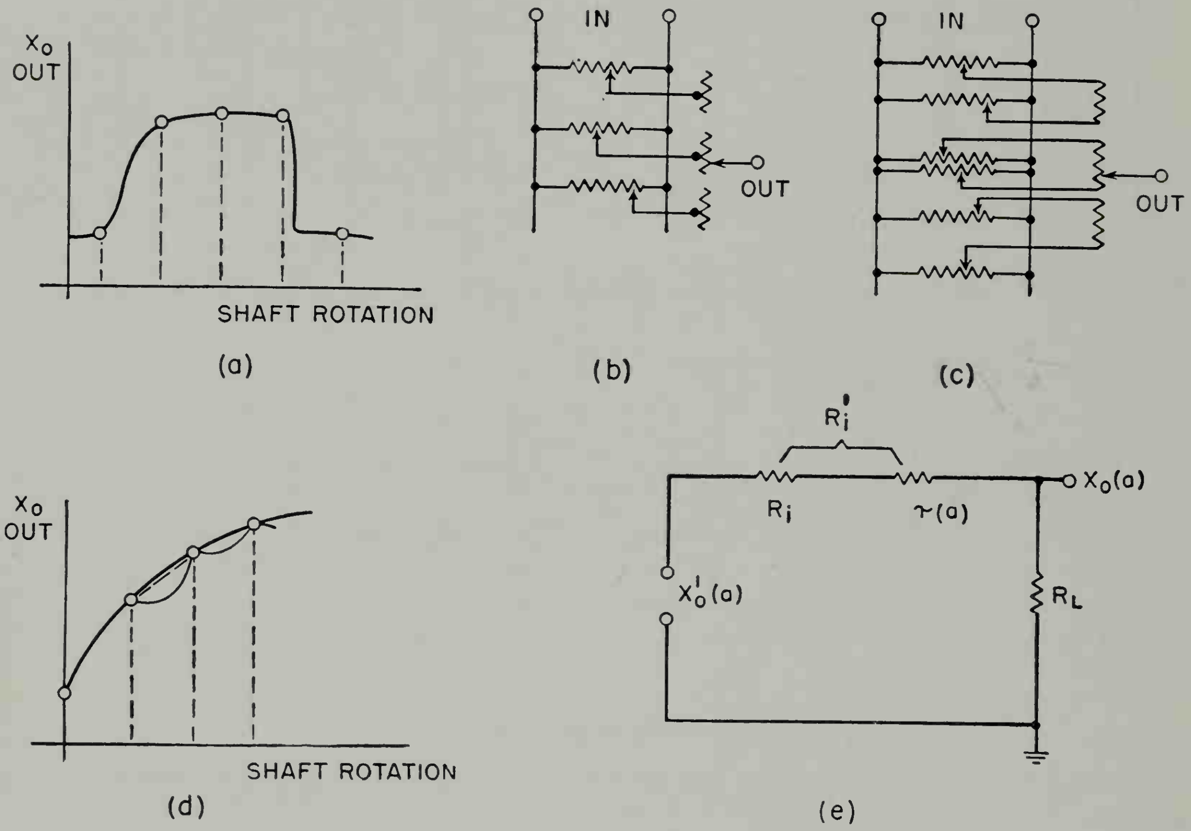

Fig. 6.39. Design considerations.

1. *It must be possible to establish the desired voltages at the tap points, with the load connected and without exceeding the power ratings of any of the resistance elements.* Any of the arrangements of Fig. 6.38 may be made to satisfy this requirement if the derivative (slope) of the function to be generated does not change too much over the interval of any one tap segment. This question may be decided very simply by drawing a rough graph of the desired function with the tap points indicated, as in Fig. 6.37.

2. *If, as in Fig. 6.39a, the function value changes appreciably between adjacent taps (large derivative), care must be taken in order not to exceed the power rating of the potentiometer in question.* It is also clear that the rate at which the output voltage can change is limited by the fact that the

voltage must vary linearly over the interval of each tap segment. An increase in the number of taps will thus permit the generation of more rapidly changing functions. Potentiometers with *split tapped windings*, of which two typical examples are shown in Fig. 6.39*b* and *c*, permit the representation of *step functions* as well as better approximations of functions having steep slopes for certain values of the argument.

3. *The output voltage must vary linearly between taps within a specified percentage, with the load connected.* This requirement will be satisfied automatically by any of the circuits shown in Fig. 6.38 if the function generator in question is working into a load of very high impedance, such as a vacuum-tube amplifier. If, however, a load of finite resistance R_L is connected between the output terminal of a function potentiometer and ground, the load will draw current from the potentiometer. The resultant voltage drop in the potentiometer winding and networks tends to make the absolute value of the output voltage smaller than that corresponding to linear interpolation. Figure 6.39*d* shows a graph of output voltage vs. shaft rotation for a case with relatively low load resistance and illustrates an extreme instance of the loading effect.

The proper design of the tapped potentiometers and adjustment networks involves the determination of the minimum load resistance in relation to the other resistance elements, which will still yield tolerably linear interpolation. For the analysis of this basic design problem, use is made of the fact that Thevenin's theorem[1] permits the representation of the tapped potentiometer and its associated networks by the *equivalent circuit* of Fig. 6.39*e*.

Let X_o' be the open-circuit voltage measured between the potentiometer output terminal and ground if the load resistance R_L is removed. According to Thevenin's theorem, the potentiometer circuit is then equivalent to a voltage source of voltage X_o' connected in series with the resistance R_i' seen by the load if all real voltage sources are replaced by short circuits.

Consider now the variation of the internal resistance R_i' and the open-circuit voltage X_o' as the sliding contact is moved through any one segment of the tapped winding. Let a denote the fraction of the segment in question traversed by the wiper, with $a = 0$ at that tap for which the value of R_i' is lower. It is possible, for each winding segment, to consider R_i' as the sum of a fixed term R_i and a term $r(a)$ variable with the shaft rotation measured by a. Thus,

$$R_i' = R_i + r(a) \qquad \text{with } r(a) \geq 0 \qquad r(0) = 0 \qquad 0 \leq a \leq 1 \quad (6.28)$$

---

[1] Smythe, W. R., *Static and Dynamic Electricity*, p. 129, McGraw-Hill, New York, 1939.

If the tapped potentiometer is a linear one, it is, moreover, clear that the open-circuit voltage X'_o must be a linear function of a:

$$X'_o(a) = (1 - a)X'_o(0) + aX'_o(1) \tag{6.29}$$

With reference to Fig. 6.39*c*, Ohm's law yields the output voltage X_o of the function potentiometer with the load connected, *viz.*,

$$\begin{aligned} X_o = X'_o \frac{R_L}{R'_i + R_L} &= X'_o(a) \frac{R_L}{R_i + R_L + r(a)} \\ &= X'_o(a) \frac{R_L}{R_i + R_L} \left[1 - \frac{r(a)}{R_i + R_L + r(a)} \right] \end{aligned} \tag{6.30}$$

Since X'_o varies linearly with a, it is seen that, for reasonably linear interpolation, r will have to be made small compared with $R_i + R_L$. This will now be discussed on a quantitative basis. It will be remembered that the tap voltages are adjusted with the load connected, so that the output voltage assumes the correct values for $a = 0$ and $a = 1$.

Let $X_{o(\text{lin})}$ be the result of a correct linear interpolation between the two tap voltages $X_o(0)$ and $X_o(1)$

$$X_{o(\text{lin})} = (1 - a)X_o(0) + aX_o(1) \tag{6.31}$$

where a is the fraction of the resistance traversed by the wiper between the taps having the voltage $X_o(0)$ and the voltage $X_o(1) \geq X_o(0)$. It may be shown[1] that an upper bound for the error due to the load is given by

$$\begin{aligned} |\epsilon| &= |X_o - X_{o(\text{lin})}| \\ &= \frac{R_L}{R_L + R_i} \left| aX'_o(1) \frac{r(1)}{R_L + R_i + r(1)} - X'_o(a) \frac{r(a)}{R_L + R_i + r(a)} \right| \\ &\leq \left| \frac{2R_L R_s}{(R_L + R_i)^2} X'_{\max} \right| \end{aligned} \tag{6.32}$$

where $X'_{\max}$ is the larger of the two quantities $|X'_o(0)|$ and $|X'_o(1)|$, and where R_s, which must be less than $R_i + R_L$, is the resistance of a winding segment between taps. Equation (6.32) serves as an estimate of the error due to the load.

The error given by Eq. (6.32) will be in addition to the interpolation error mentioned above. It may be compared to the full source voltage X_i or, more conservatively, to the tap voltage

$$(R_L/R_L + R_i)X'_{\max}$$

In the latter case the percentage error ϵ is given by

$$\frac{1}{2}\epsilon \leq \frac{100R_s}{R_L + R_i} \tag{6.33}$$

---

[1] Eq. (6.32) corrects an error in Korn, *op. cit.*

The use of Eq. (6.33) as a convenient design formula is illustrated by the following example. For all the arrangements shown in Fig. 6.38, the lowest possible value $R_{\min}$ of R_i is $R_i = 0$, so that (6.33) yields

$$\frac{1}{2}\epsilon \leq \frac{100R_s}{R_i + R_L} \leq \frac{100R_s}{R_L} \tag{6.34}$$

It should be kept clearly in mind that (6.34) is *not* a necessary condition which applies to all special cases of function generating potentiometers; it is merely a rather general *sufficient* condition which will hold for many different circuits and a wide range of functions to be generated. If it should be desirable to find optimum design values for special cases, and in particular for the representation of one fixed function, Eq. (6.32) should be referred to for each winding segment of the potentiometer in question. In many cases, such a procedure will lead to much higher values of the ratio R_s/R_L than those given by (6.34).

In practically all cases the loading errors, as given by (6.32), may be reduced by introducing resistors in series with either the load or the voltage source supplying the function potentiometer. This will increase R_L or R_i, respectively, at the expense of the output level. *A better scheme is to introduce series resistors between certain taps and the corresponding adjustment networks.* This procedure decreases only the maximum voltage available at the taps in question and may also help to increase the input impedance of the potentiometer and to prevent excessive currents in the winding segments.

4. *The function generator should present a specified impedance to the voltage source.* Since the formulas (6.32) to (6.34) for the loading error all involve resistance *ratios* referred to the given load resistance R_L, the input impedance is determined by the network arrangement chosen and by the load resistance R_L. The choice of networks has been discussed in the preceding paragraphs; it may, however, not be possible to obtain input impedances as high as required if the load resistance R_L is necessarily too low. In such cases it may be necessary to sacrifice the output level in order to isolate the function generator from either the source or the load by means of series resistors. Vacuum-tube amplifiers will, of course, serve the same purpose without sacrificing output level.

Construction of Tapped Potentiometers. At the present time, several manufacturers of wire-wound potentiometers will tap their windings on special order, but for small production runs and large numbers of taps the cost of this operation may be quite high. If the demand for tapped potentiometers should increase, however, there is little reason why the cost per tap should exceed 50 cents on orders of 25 or more potentiom-

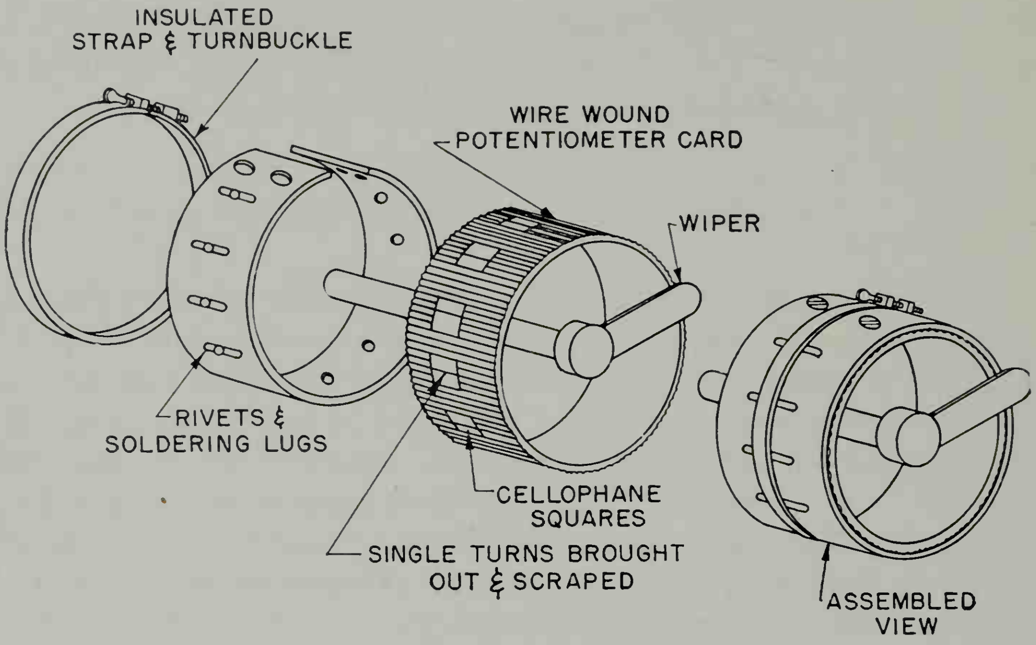
INSULATED STRAP & TURNBUCKLE
WIRE WOUND POTENTIOMETER CARD
WIPER
RIVETS & SOLDERING LUGS
CELLOPHANE SQUARES
SINGLE TURNS BROUGHT OUT & SCRAPED
ASSEMBLED VIEW

Fig. 6.40. Exploded view of tapped potentiometer.

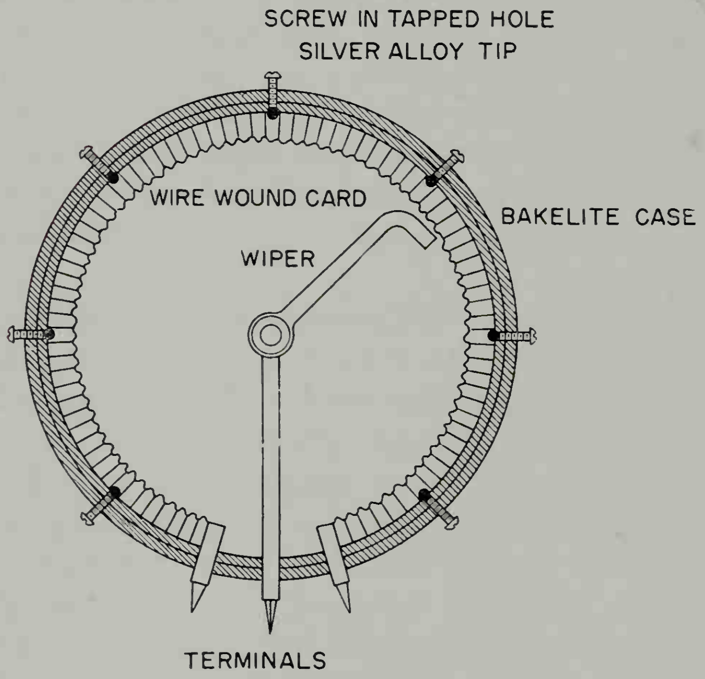

Fig. 6.41. Alternative method for tapping.

eters. In any case, the operations involved in tapping potentiometers in the laboratory are not too difficult and now will be described.

Figures 6.40 and 6.41 illustrate two possible methods for tapping several types of wire-wound potentiometers. At this writing, only wire-wound potentiometers were considered to have the resolution necessary for adaptation as universal function generators.

Figure 6.40 shows an exploded view of a General Radio type of potentiometer which has been modified by tapping. This type of potentiometer has a cylindrical card or mandrel wound with resistance wire. At each desired tap location, part of a single turn of wire has been lifted off the card. This wire was then separated from the rest of the winding by an insulating strip of plastic or cellulose tape and scraped free of insulation. Contact to these free portions of the winding is then made by (preferably silver-plated) rivets or screwheads fastened to the outer bakelite housing of the potentiometer in the proper locations.

It is also possible to place a thin piece of copper foil between the potentiometer wire and insulating strip at the point of tapping. Soft solder is then applied and should form a tight bond between the wire and the copper foil. Finally, the lead wire is soldered directly to the copper foil.[1] It can be seen from Fig. 6.40 that the thin outer potentiometer housing is attached to the inner shell by screws which also serve to hold the card in place. Since these screws alone may not effect sufficiently firm contact between the rivets and the winding, it is suggested that an insulated strap, also shown in Fig. 6.40, be fitted over the rivets and secured tightly by means of a turnbuckle.

Figure 6.41 shows an alternative method for tapping wire-wound potentiometers and is suited to potentiometers having a thick outer or inner bakelite shell. The card and winding are first removed from the shell. Holes corresponding to the desired tap locations are drilled in the shell; it will be remembered that the location of the taps is not critical. The holes are then tapped with a fine thread. After the winding has been replaced in the shell, it may be contacted by silver- or gold-tipped screws screwed firmly into the tapped holes. Wires may be soldered to these screws, after which the screwheads are secured in position by a layer of sealing wax.

This last method of tapping is especially applicable to potentiometers of the helical-card type (Helipots), some of which have shells with thick ribs admirably suited for drilling and tapping. The proper location of the holes for such a helical potentiometer is easily found by placing the shell in a lathe and cutting a very light thread of the proper pitch. This thread, which need only barely score the ribs in the shell, may be located so as to indicate accurately the location of the helical winding inside the shell. The 40-tap function potentiometer shown in Fig. 6.43*a* was constructed in this manner, and the locating scores may still be seen on the ribs.

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[1] Private communication from R. J. Sullivan, Chief, Potentiometer Research Unit, Air Force Cambridge Laboratories, who was kind enough to point this out to the author.

Figure 6.42 illustrates one of several methods of locating the adjustment potentiometers and networks around, above, or beside a tapped potentiometer. The most accessible location of the adjustment shafts may have to be chosen separately for each application.

A more sophisticated arrangement is applicable wherever shortness of leads is not a paramount consideration and is illustrated by Fig. 6.43*a*, *b*, and *c*. The leads from the potentiometer taps are brought out to a cable connector (Fig. 6.43*a* and *b*). The adjustment networks are arranged on a separate panel fixed with a multiple connector in such a manner that it can be either plugged directly onto the potentiometer or connected to it by a suitable cable. Such an arrangement is shown in Fig. 6.43*c*; note the shaft locks on the adjustment potentiometers. Such arrangements permit the interchanging and filing of several function networks for each tapped potentiometer. This makes it possible to set up several frequently needed function networks once and for all and to interchange them at will. Since the tapped potentiometer proper constitutes a major part of the total cost of the function generator, this may result in considerable savings. This arrangement also permits more convenient location of the adjustment shafts (*e.g.*, on the front panel of a computing machine) for easier adjustments and function changes. A computer installation comprising function generating potentiometers of this type is shown in Fig. 8.7 and described in Sec. 8.7.

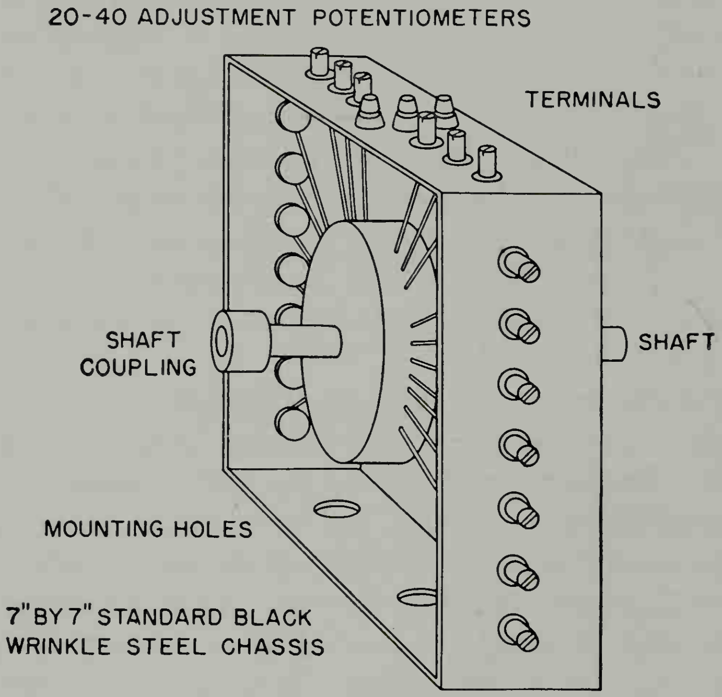

20-40 ADJUSTMENT POTENTIOMETERS
TERMINALS
SHAFT COUPLING
SHAFT
MOUNTING HOLES
7" BY 7" STANDARD BLACK WRINKLE STEEL CHASSIS

Fig. 6.42. Arrangement of adjustment shafts.

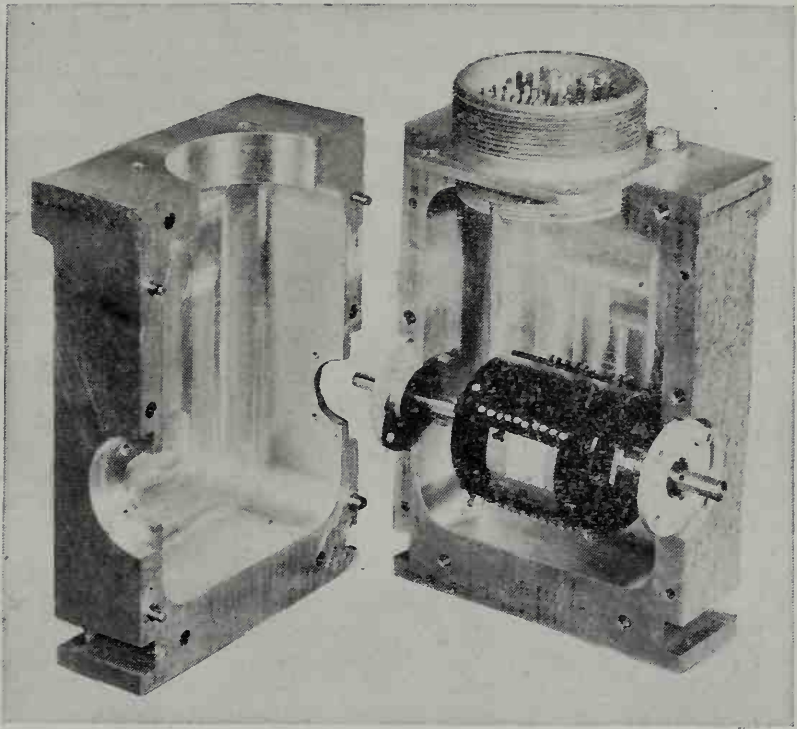


FIG. 6.43*a*. Photograph of a forty-tap helical-type potentiometer (opened).

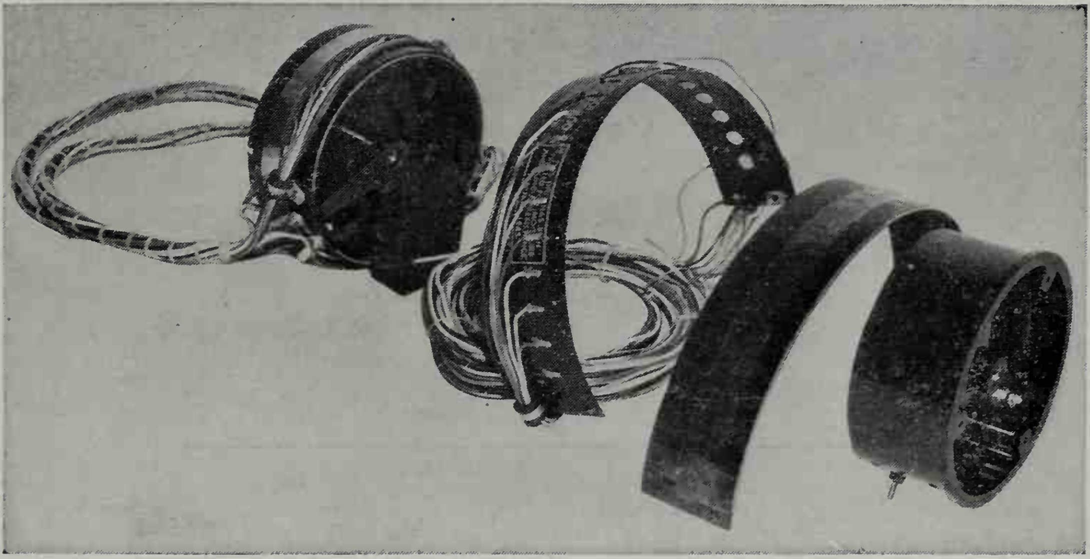


FIG. 6.43*b*. Exploded view of function potentiometer.

6.7. Limiters and Other Function Generators Using Diode Characteristics. *Limiters.* Perhaps the simplest function generators used in d-c analog computers are *limiters* which make use of the characteristics of nonlinear circuit elements to obtain an output voltage X_o, say, which varies with the input voltage X_1 in a specified manner. The input vs. output voltage characteristic of an ideal limiter is shown in the solid lines of Fig. 6.44. Such a characteristic is particularly useful for the solution of certain control problems which require a machine variable X_o to be limited to the range between two values $X_{\min}$ and $X_{\max}$. In problems concerning aircraft control design, for example, the machine variable X_o may correspond to a control surface deflection limited by stops on either side of its neutral position (see also Secs. 3.2 and 3.3). Limiters may also be useful for protecting certain measuring and recording devices used in computers from overloads.

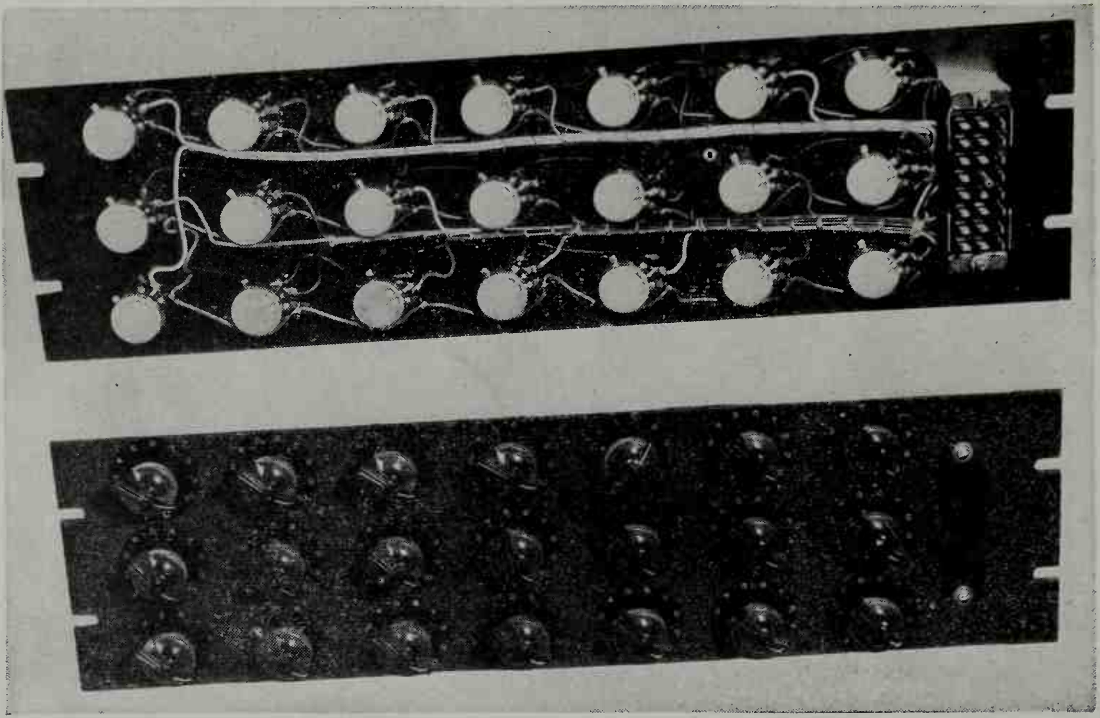


FIG. 6.43*c*. Front and back views of a panel bearing adjustment potentiometers for a twenty-tap function potentiometer.

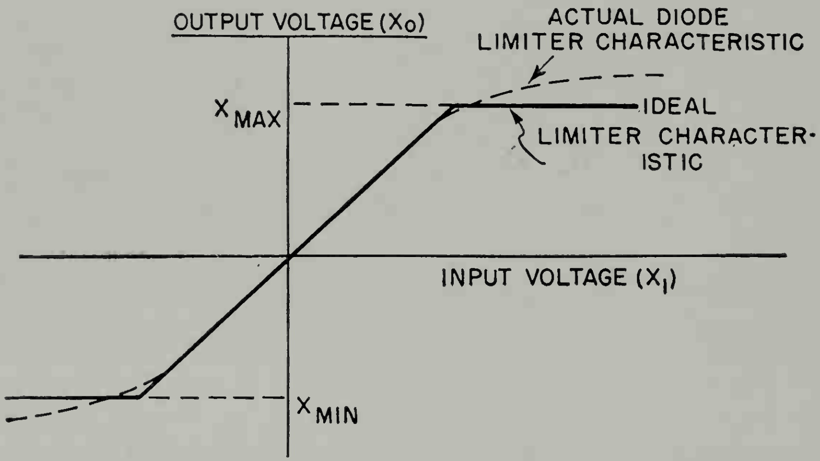

OUTPUT VOLTAGE (Xo)
ACTUAL DIODE LIMITER CHARACTERISTIC
X MAX
IDEAL LIMITER CHARACTERISTIC
INPUT VOLTAGE (X$_I$)
X MIN

FIG. 6.44. Limiter characteristics.

The most frequently used limiting circuit is shown in Fig. 6.45*a*. A *diode rectifier* will conduct if and only if its plate voltage is higher than its cathode voltage. Accordingly, diode V_1 will conduct if and only if the output voltage X_o of the limiter becomes larger than the bias voltage $X_{\max}$. Similarly, diode V_2 will conduct if and only if X_o is less than the bias voltage $X_{\min}$. The bias voltages $X_{\max}$ and $X_{\min}$ may be obtained from standard voltage sources in the computer through potentiometers (see Fig. 6.45*a*) which permit convenient adjustment of the limiting levels.

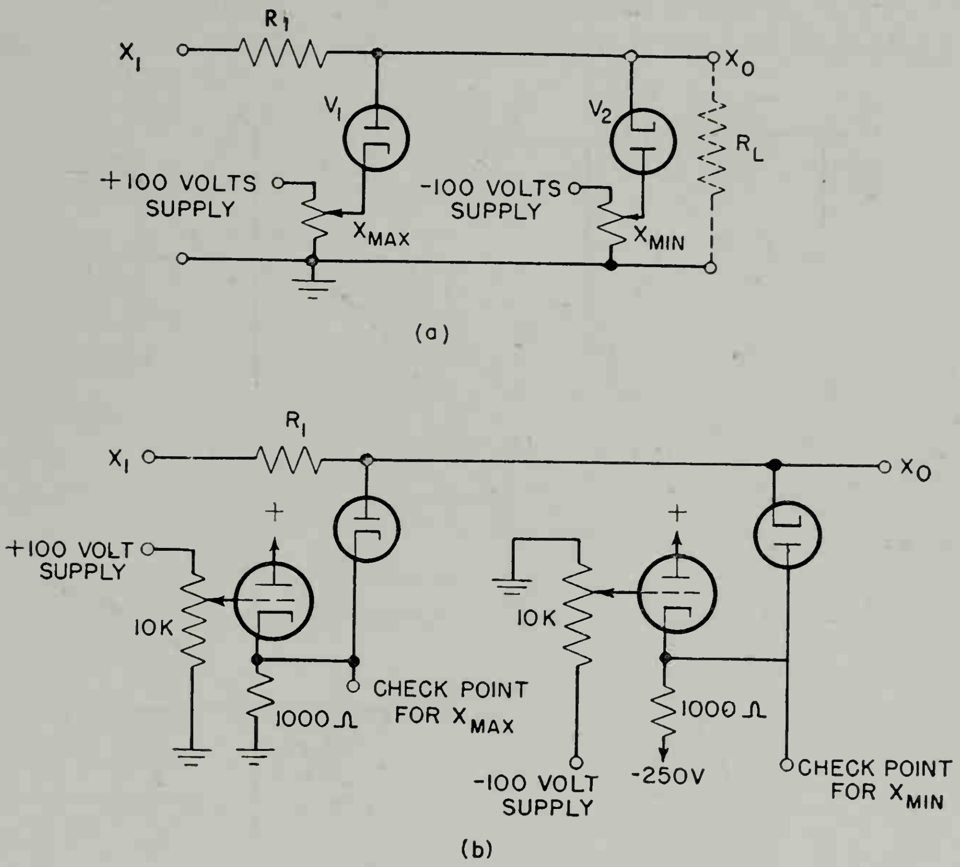


FIG. 6.45. Parallel-diode limiter circuits.

When the diodes are not conducting, the performance equation of the circuit of Fig. 6.45*a* is

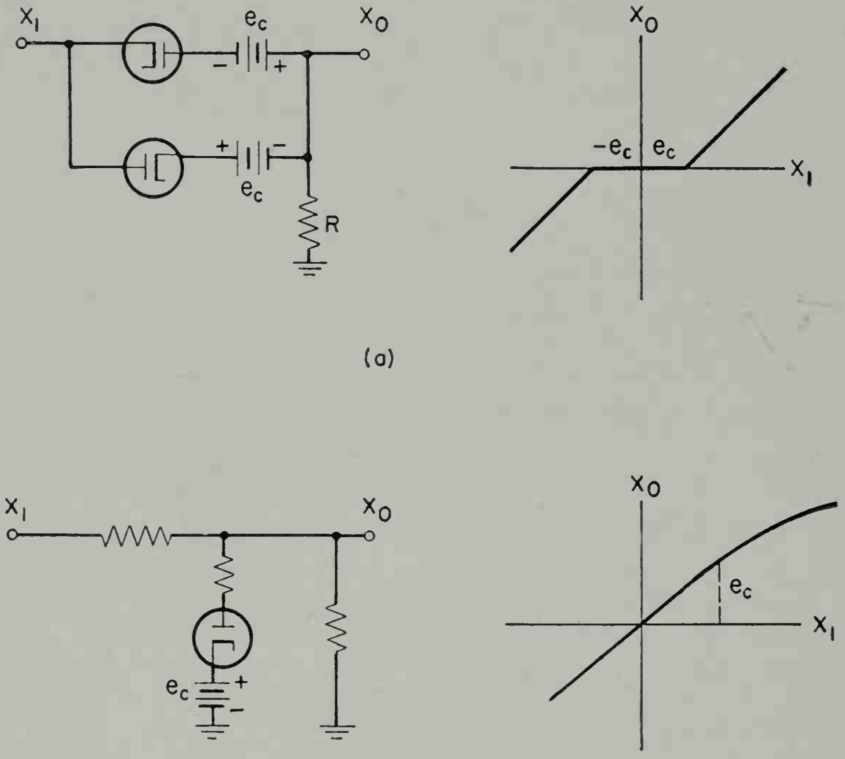
$$X_o = \frac{R_L}{R_L + R_1} X_1 \tag{6.35}$$

where the load resistance R_L is usually the input resistance of an operational amplifier following the limiter (see, for instance, Sec. 3.2). When one of the diodes is conducting, the performance equation becomes

$$X_0 = \frac{R_L}{R_L r + R_1 r + R_1 R_L} (X_1 r + e_c R_1)$$
$$(\approx e_c \text{ if } r \ll R_1 \qquad r \ll R_L) \tag{6.36}$$

where r is the combined internal resistance of one limiter diode and the bias voltage supply including the adjusting potentiometer, and e_c is the positive or negative bias voltage.

This arrangement will result in a limiter characteristic similar to that shown in dash lines in Fig. 6.44. The slope of this characteristic is approximately given by Eqs. (6.35) and (6.36); the smaller the ratios r/R_L and r/R_1, the more closely the curve will approach the ideal limiter characteristic, shown in solid lines. R_1 and R_L are usually between 0.2 and 2 megohms; the value of r may be as large as 5,000 to 10,000 ohms in the circuit of Fig. 6.45a. This value may be reduced to less than 800 ohms by using cathode-follower circuits like those shown in Fig. 6.45b for setting the bias levels. The internal resistance of the conducting diode itself is of the order of a few ohms or less; either vacuum or crystal diodes may be used.



(a)

(b)

FIG. 6.46. Two nonlinear networks and their transfer characteristics.

The transition between the ON and OFF or conducting and nonconducting conditions of a diode may not be so sharp as desirable for very accurate computations; but generally speaking, the characteristic of a parallel diode limiter may be made to approximate the ideal limiter characteristic within 0.5 volt, which is sufficient for most applications. Sharper limiting is possible with *limiting amplifiers* in which the plate or grid cutoff characteristics of properly biased triodes and pentodes are used for "clipping." The output voltage of such a limiting amplifier is limited much like that of any overdriven amplifier.

The Synthesis of Other Function Generators Based on Diode Circuits. A vacuum or crystal diode may be regarded as a switch which will either open or close a circuit depending on the applied voltage. The properties of a network containing diodes will, then, in general vary with the applied voltages. Figure 6.46 shows examples of simple "nonlinear" circuits comprising diodes. When the respective positive or negative voltages across the circuit in Fig. 6.46*a* or *b* exceed a given bias level e_c, the circuits change their properties in two ways:

1. The resistance is decreased from a very large value to approximately R_1.
2. The bias voltage e_c appears in series with the applied voltage E.

The usefulness of such circuits for the purpose of generating a nonlinear function of an input voltage was demonstrated in the limiter circuits discussed above. *By combining several diode circuits of the type shown in Fig. 6.47 with linear networks and/or feedback amplifiers, it is possible to generate output voltages which approximate almost any desired function of an input voltage.*

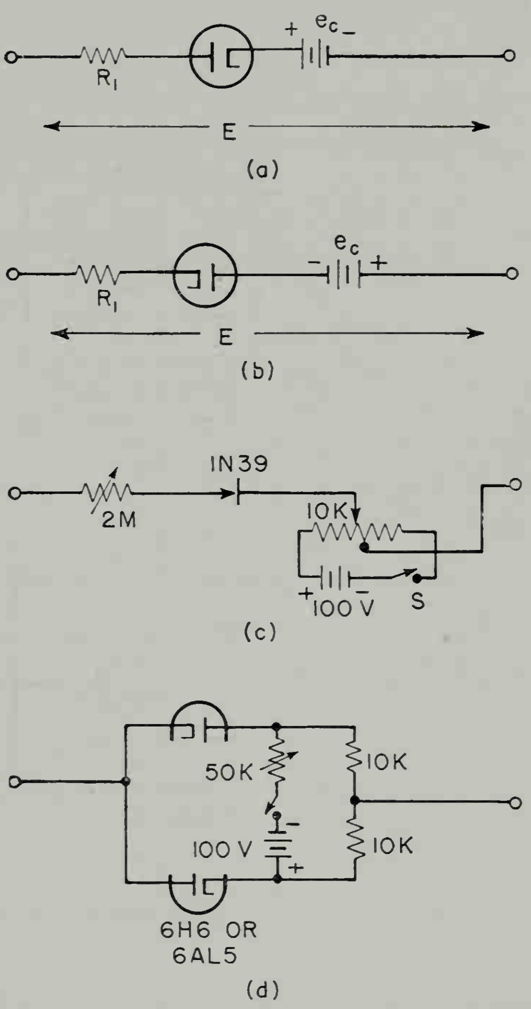


FIG. 6.47. Nonlinear circuit elements based on diode characteristics.

The simple circuit of Fig. 6.46*a* is an example of a diode function generator which, like the diode limiter, is frequently used in representing the properties of dynamical systems. The output vs. input voltage characteristic (transfer characteristic) of this nonlinear network is also shown and is similar to the relation between the restraining force and the displacement of a mass restrained by springs with a dead space for zero displacement (see also Sec. 3.3). The reader should establish how the voltage characteristic shown in Fig. 6.46*a* would be changed through the addition of resistors in series and in parallel with each of the diodes.

Figure 6.46*b* shows another example of a nonlinear network and its transfer characteristic.

Nonlinear Operational Amplifier Circuits. It was shown in Chap. 1 that the use of feedback amplifier circuits instead of simple networks will permit greater flexibility in the choice of the voltage and impedance levels employed in computing elements.

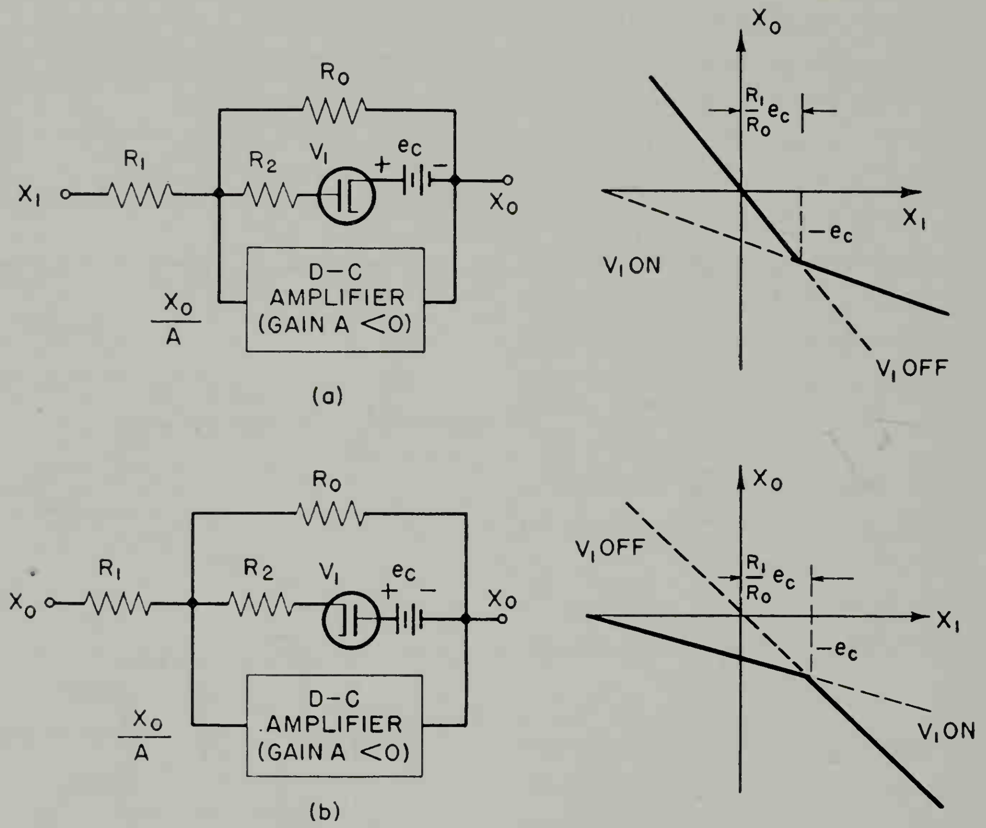


FIG. 6.48. Illustrating the operation of a nonlinear operational amplifier.

Figures 6.47 to 6.49 show a number of operational amplifier circuits (see Chap. 4) using nonlinear feedback networks to obtain various nonlinear transfer characteristics. An analysis of the relatively simple circuits of Fig. 6.48 will serve to describe the operation of other similar circuits.

In each of the two circuits shown in Fig. 6.48, the circuit performance is given by the nodal equation

$$\left(\frac{X_o}{A} - X_1\right)\frac{1}{R_1} + \left(\frac{X_o}{A} - X_o\right)\frac{1}{R_o} = 0 \tag{6.37}$$

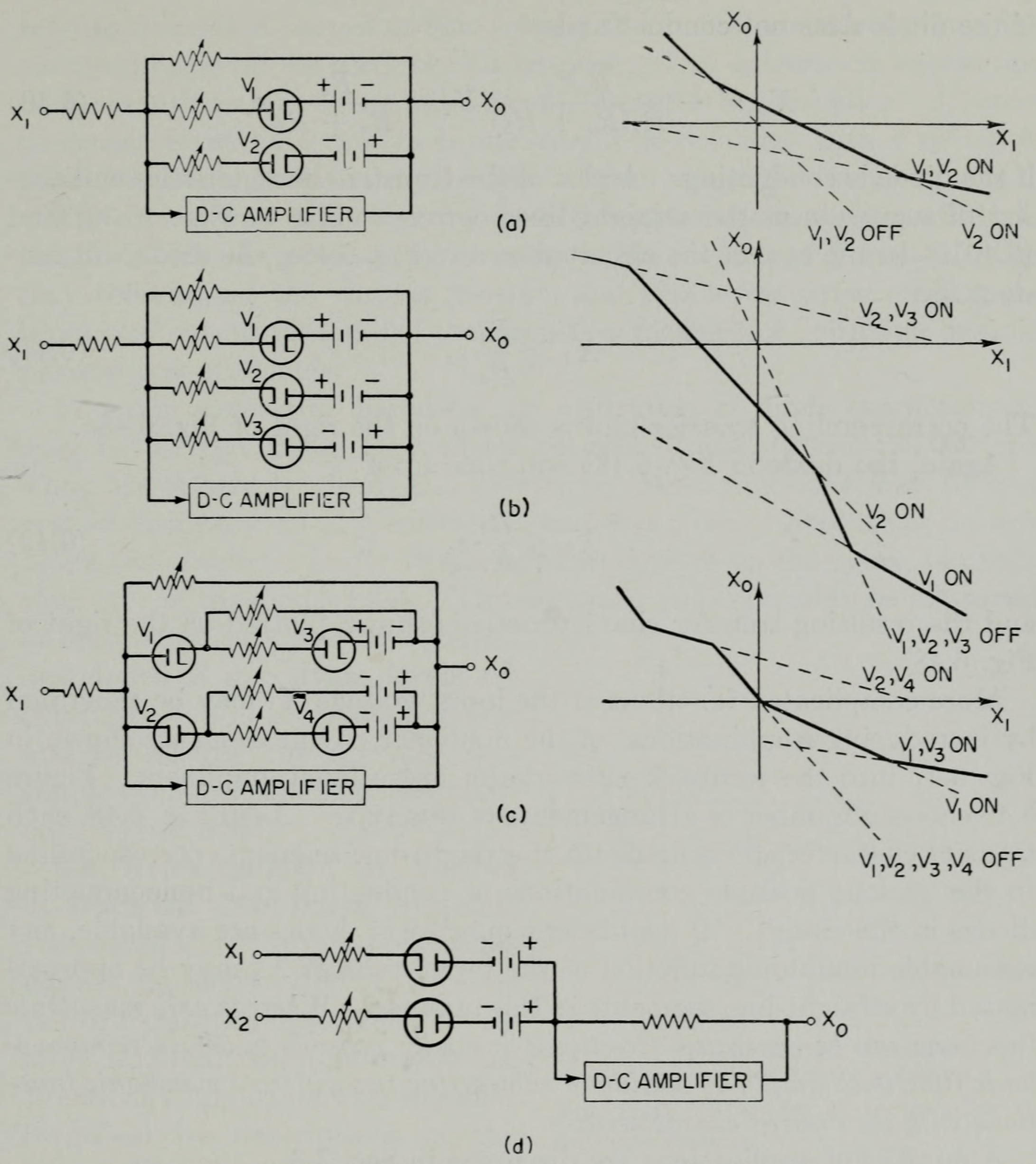


FIG. 6.49. Nonlinear operational amplifier circuits.

as long as the diode does not conduct. Whenever the diode does conduct, the nodal equation becomes

$$\left(\frac{X_o}{A} - X_1\right)\frac{1}{R_1} + \left(\frac{X_o}{A} - X_o\right)\frac{1}{R_o} + \left(\frac{X_o}{A} - X_o - e_c\right)\frac{1}{R_2} = 0 \qquad (6.38)$$

For large absolute values of the amplifier gain A, it will be permissible to neglect X_o/A in Eqs. (6.37) and (6.38); the transfer characteristics of the circuit shown in Fig. 6.48a are then simply

$$X_o = -\frac{R_o}{R_1}X_1 \qquad (6.39)$$

if the diode does not conduct and

$$X_o = -\frac{R_oR_2}{R_1(R_o + R_2)}X_1 - \frac{R_o}{R_o + R_2}e_c \tag{6.40}$$

if the diode is conducting. A plot of the transfer characteristics will consist of segments of the straight lines corresponding to Eqs. (6.39) and (6.40). In the case of the circuit shown in Fig. 6.48*a*, the diode will conduct for

$$X_1 \geq \frac{R_1}{R_o}e_c \tag{6.41}$$

The corresponding transfer plot is shown on the right of Fig. 6.48*a*.

Again, the diode in Fig. 6.48*b* will conduct if

$$X_1 \leq \frac{R_1}{R_o}e_c \tag{6.42}$$

and the resulting transfer characteristic is shown plotted on the right of Fig. 6.48*b*.

More complicated functions of the input voltage X_1 may be generated by introducing combinations[1] of the nonlinear circuit elements shown in Fig. 6.47 into the feedback networks of operational amplifiers. Figure 6.49 shows a number of arrangements of this type. As in Fig. 6.48, each transfer characteristic is made up of straight-line segments corresponding to the various possible combinations of conducting and nonconducting diodes in the circuit. If a sufficient number of diodes are available, any reasonable monotonic function of the input voltage X_1 may be approximated by straight-line segments in this manner. *Whereas only monotonic functions can be generated directly, it is easily possible to obtain nonmonotonic functions simply by adding or subtracting two different monotonic functions with the desired characteristics.*

A number of applications are discussed in Sec. 3.3.

Practical Circuits. Most vacuum or crystal diodes may be used as rectifier elements for computing purposes if their current and voltage ratings are not exceeded. *In practice, a d-c analog computer may be provided with "isolated rectifier elements" of the types shown in Fig.* 6.47*c and d, which will serve to synthesize the various types of nonlinear computing elements just discussed. It is clear that such devices are more generally useful than the simple diode limiters incorporated in most commercial d-c analog computers.* Unlike the simple limiters, the isolated rectifier elements will require *individual sources of bias voltage;* the common computer supplies cannot, in general, be used for this purpose. Dry batteries in connection with potentiometer voltage adjusters are useful as bias supplies. In order to conserve the batteries, each one should be provided with a switch or relay for opening the battery circuit between computer runs.

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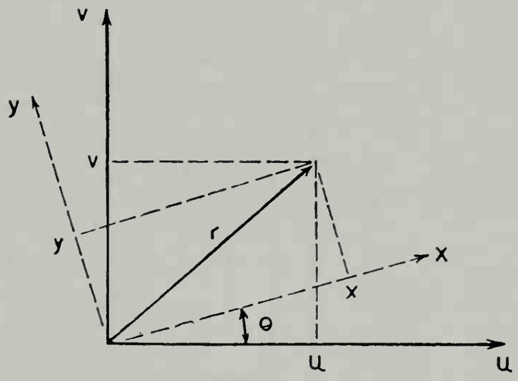
[1] McCann, G. D., C. H. Wilts, and B. N. Locanthi, Application of the California Institute of Technology Electric Analog Computer to Non-Linear Mechanics and Servomechanisms, *Trans. AIEE* (T9165), **68**, 1949.

The easiest way of obtaining a desired transfer characteristic will be to establish only the general circuit setup in the manner of Fig. 6.49. One may then adjust the various resistors and bias levels while reading the input and output voltages of the circuit by means of a voltmeter or comparison potentiometer.

Function generators based on the utilization of diode characteristics may be used at signal frequencies between 0 and at least 20,000 cps. They are relatively cheap and compact. Most functions may be generated with an accuracy between 1 and 5 per cent,[1] depending on how much the specific transfer characteristics depend on the aging, temperature, etc., of the diodes used. This accuracy might possibly be improved, at least in special cases, by a judicious choice of impedance levels, by careful selection of crystal diodes, and by placing the crystals in a simple constant-temperature chamber (crystal oven).

6.8. Representation of Vectors in Different Coordinate Systems.

Introduction. Many applications require the description of the same vector quantities in terms of two or more different coordinate systems. Practically important examples are furnished by descriptions of the forces acting on an aircraft. These forces (drag, thrust, lift, gravity, etc.) may be described in terms of coordinate axes fixed with respect to the earth, the aircraft, or the relative wind (see also Chap. 3). A study of the transformations of the components of *two-dimensional* vector quantities will illustrate all the necessary analog-computer techniques used for such transformations.



$$X = u \cos \theta + v \sin \theta$$
$$Y = - u \sin \theta + v \cos \theta$$

FIG. 6.50. Rotation of rectangular coordinate system.

Representation of a Vector in Different Coordinate Systems: Rotation of Coordinate Axes. Figure 6.50 shows a vector which may be described by its components x,y along the x,y coordinate axes as well as by the components u,v along the u,v coordinate axes. Both the x,y and u,v reference frames are rectangular Cartesian coordinate systems having a common origin; the angle θ between the x axis and the u axis is defined as shown in Fig. 6.50. Under these conditions, the vector components x,y are related to the components u,v by the *transformation equations*

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[1] *Ibid.*

$$\left.\begin{aligned} x &= u\cos\theta + v\sin\theta \\ y &= -u\sin\theta + v\cos\theta \end{aligned}\right\} \qquad (6.43)$$

If machine variables X, Y, U, V corresponding to the vector components x, y, u, v, respectively, are chosen with equal scale factors a so that

$$X = ax \qquad Y = ay \qquad U = au \qquad V = av \qquad (6.44)$$

the *machine equations* corresponding to (6.43) are simply

$$\left.\begin{aligned} X &= U\cos\theta + V\sin\theta \\ Y &= -U\sin\theta + V\cos\theta \end{aligned}\right\} \qquad (6.45)$$

If the angle θ can be represented by a servo shaft displacement equal to θ, the relations (6.45) can be established in a d-c analog computer in the manner shown in Fig. 6.51. The multiplications by trigonometric functions are performed by means of function potentiometers on the θ shaft, and summing amplifiers are used for addition.

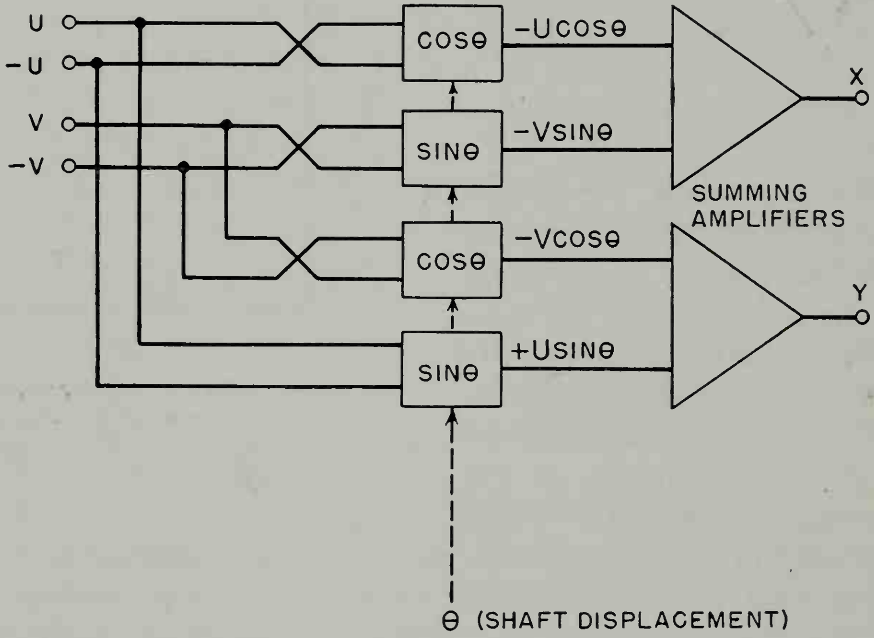

FIG. 6.51. D-c analog-computer setup for Eq. (6.45).

The resistance windings, or "cards," of commercially available sine and cosine potentiometers usually cover only 180 deg of shaft rotation. Figure 6.52*a* shows how such sine or cosine potentiometers may be connected up to multiply a voltage by trigonometric functions of angles in the first and second quadrants (from 0 to 180 deg). By connecting such potentiometers into the computer circuit in the manner shown in Fig. 6.52*b*, input voltages can be multiplied by sine and cosine functions of angles between minus and plus 90 deg.

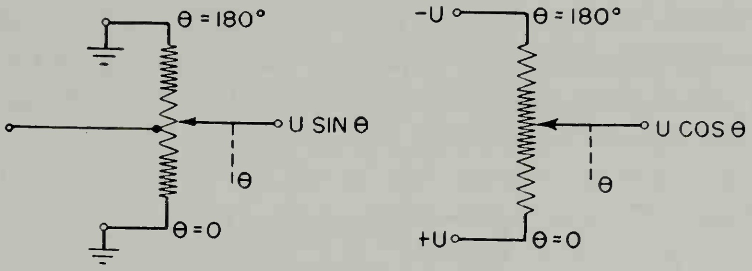

FIG. 6.52*a*. Connections for sine and cosine potentiometers for operation between 0 and 180 deg. Note the two types of tapered windings.

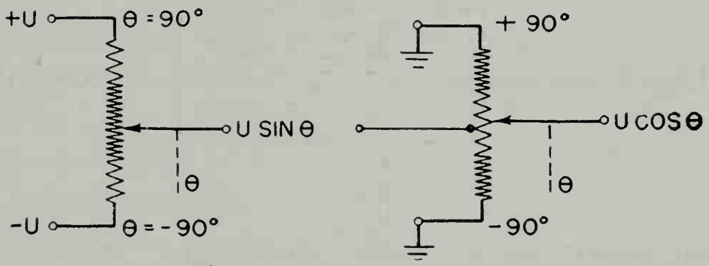

FIG. 6.52*b*. Connections for sine and cosine potentiometers for operation between minus 90 to plus 90 deg. Note the two types of tapered windings.

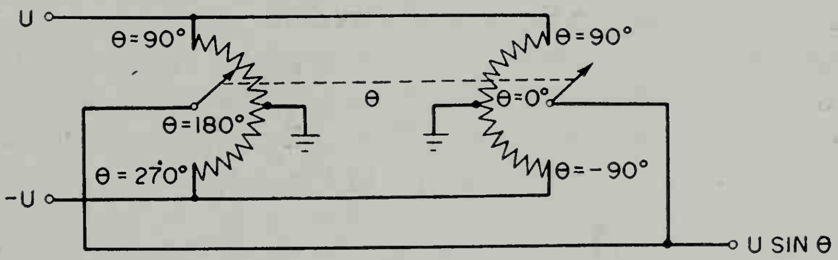

FIG. 6.52*c*. Example of potentiometer connections for continuous rotation. Each potentiometer card is used for alternate 180 deg of angular displacement. Integral multiples of 360 deg may be added to each of the angular displacements indicated in the diagram.

If, as is frequently the case, the range of the angle θ exceeds 180 deg, two sine potentiometers in the circuit of Fig. 6.52*c* may be used. Both potentiometer wipers can rotate continuously. They are electrically connected and mechanically ganged in such a manner that one of the potentiometers is used for values of the angle θ in the fourth and first quadrants and the other one for values of θ in the second and third quadrants.

A more convenient function generator suitable for continuous rotation is the *rectangular card potentiometer* shown schematically in Fig. 6.53. The rectangular card potentiometer[1] not only permits 360-deg rotation but also yields two output voltages; one output voltage is proportional to the sine and the other to the cosine of the potentiometer shaft displacement. As shown in Fig. 6.53, a rectangular card potentiometer consists of a closely wound rectangular resistance winding or card rotated by the input shaft and four contacts or wipers attached to the fixed potentiometer case. As the card is rotated, each contact effectively travels over the card in a circle about the axis of rotation.

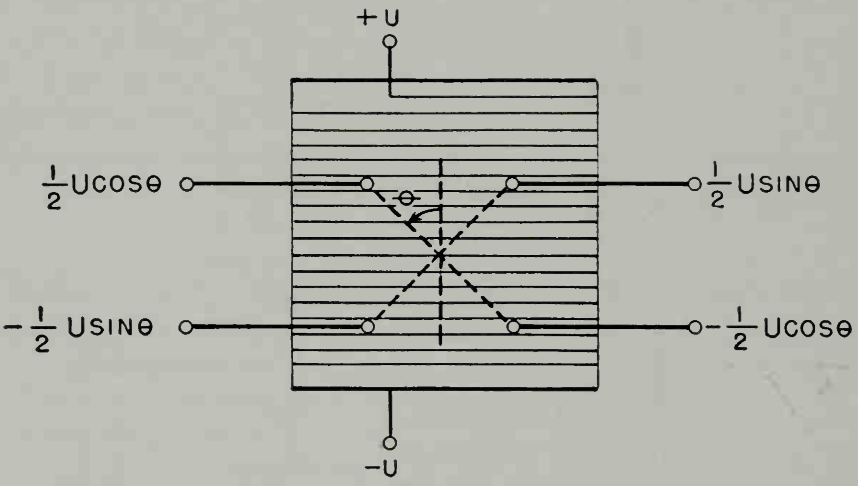

FIG. 6.53. Rectangular card sine-cosine potentiometer. The axis of rotation is perpendicular to the plane of the diagram.

The voltage at each contact will be proportional to the sum of the voltage at the center of the card and a sinusoidal voltage of the angular displacement of the card. The two voltage differences between the pairs of diametrically opposite contacts shown in Fig. 6.53 will be proportional to the sine and cosine of the angle or rotation, since the center voltage will be canceled out on taking the difference. Each voltage difference may be conveniently obtained by means of a differential amplifier (see Sec. 5.3). As long as a differential output circuit of this kind is used, the axis of rotation need not even pass through the exact geometrical center of the contacts,[2] since any off-center errors will be canceled out. The use of differential output circuits of some type[1] with rectangular card potentiometers is recommended in all computer applications.

---

[1] The rectangular card potentiometer was designed at the MIT Radiation Laboratory, Cambridge, Mass., and is now commercially available from the Gamewell Company, Newton Upper Falls, Mass. A detailed description of this device will be found in Greenwood, Holdam, and MacRae, *op. cit.*, pp. 111–116.

[2] For a detailed discussion of the problems involved in manufacturing rectangular-card potentiometers, see P. Rosenberg, Sinusoidal Potentiometers Types RL10CB RL10CD, RL10E, RL14, *RL Rept.* 423, Aug. 16, 1943.

Each rectangular card potentiometer can replace between two and four ordinary tapered potentiometers; the functions are generated with an accuracy of the order of 0.6 per cent of full scale. Such potentiometers are, however, not too easy to construct because of the close mechanical tolerances required, and their construction makes it difficult to gang several rectangular card potentiometers on a common shaft.

Another method for obtaining the transformations (6.45) with an analog computer involves the use of a-c computing elements. Two a-c voltages of equal frequency and phase may be modulated[2] with d-c voltages U and V, respectively. If these a-c voltages are impressed on the two stator windings of a special rotatable transformer called a *resolver* (see Fig. 6.54), whose rotor displacement is equal to θ, the rotor output a-c voltages will be modulated by the voltages X and Y given by the transformation equations (6.45). D-c voltages proportional to X and Y may then be obtained through the use of phase-sensitive demodulators.[3] Accuracies better than 0.5 per cent can be obtained with such arrangements, but the modulators and demodulators as well as the resolvers have to be designed with great care to obtain such accuracy. Resolvers will be less susceptible to mechanical wear than potentiometers, but the modulators, demodulators, and resolver driver amplifiers required with computer circuits using resolvers will add to the cost and complexity of the equipment. Such circuits are, therefore, most useful if one modulator can be used for several multiplications or if the a-c output can be used directly for driving an a-c servomechanism.

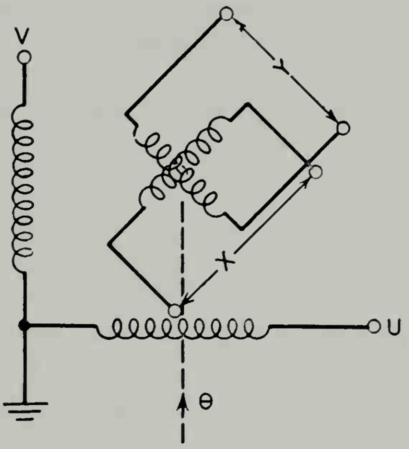

FIG. 6.54. Use of a resolver to set up Eqs. (6.45). The values indicated are amplitudes of a-c voltages.

If the computer in question works on a time scale so fast that computing servomechanisms cannot be used, the angle θ will have to be represented by a voltage in the machine, and a more complicated combination of electronic function generators and multipliers will be required. The trigonometric functions might also be generated by implicit computation (see Sec. 6.5).

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[1] A circuit somewhat different from that of Fig. 6.53 is shown in Greenwood, Holdam, and MacRae, *op. cit.*

[2] Chance *et al.*, *op. cit.*, Chaps. 12 and 14.

[3] *Ibid.*

Transformation from Polar to Rectangular Coordinates. If the polar coordinates r and θ of a two-dimensional vector are given, it is readily seen from Fig. 6.55 that the transformation equations to rectangular coordinates are

$$\left.\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}\right\} \qquad (6.46)$$

Assuming that equal scale factors are used to introduce the machine variables X, Y, and R corresponding to x, y, and r, respectively, it is possible to obtain a set of corresponding machine equations

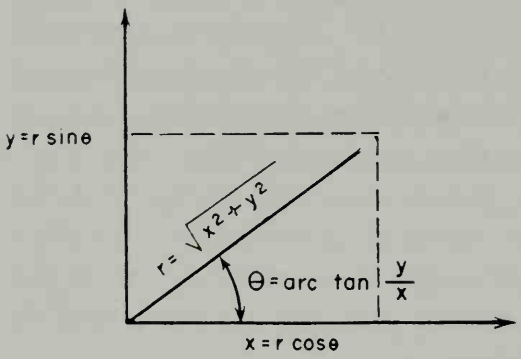

FIG. 6.55. Transformation from polar to rectangular coordinates.

$$\left.\begin{aligned} X &= R \cos \theta \\ Y &= R \sin \theta \end{aligned}\right\} \qquad (6.47)$$

If a servo shaft displacement equal to the angle θ can be made available in the computer, the multiplications indicated in the transformations (6.47) may be performed by means of function potentiometers as shown in Fig. 6.56*a* or by means of a resolver as shown in Fig. 6.56*b*. In "fast" computers, electronic multipliers and function generators will again be required to establish the transformation.

Rectangular to Polar Transformations. The equations of transformation from rectangular to polar coordinates are obtained by solving Eqs. (6.46) for r and θ. One finds

$$\left.\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \text{arc tan } \frac{y}{x} \end{aligned}\right\} \qquad (6.48)$$

If equal scale factors are used for r, x, and y, the corresponding machine equations are

$$\left.\begin{aligned} R &= \sqrt{X^2 + Y^2} \\ \theta &= \text{arc tan } \frac{Y}{X} \end{aligned}\right\} \qquad (6.49)$$

Whereas these relations could be established directly on an analog computer, it may be more convenient to obtain R and θ by *implicit solution* of the equations

$$\left.\begin{aligned} -X \sin \theta + Y \cos \theta &= 0 \\ X \cos \theta + Y \sin \theta &= R \end{aligned}\right\} \tag{6.50}$$

which are equivalent to the transformations (6.49).

The first equation (6.50) does not contain R and may be solved implicitly for θ. Assuming that θ may again be represented by a servo shaft

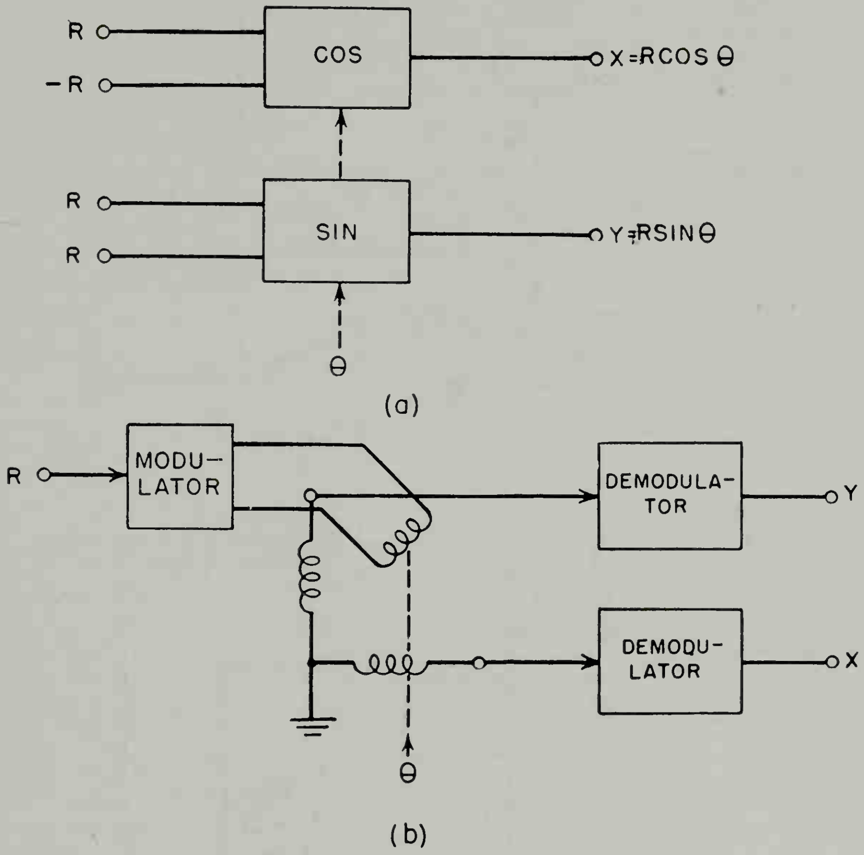

FIG. 6.56. Transformation from polar to rectangular coordinates by means of function potentiometers and resolvers.

displacement equal to θ, the implicit solution of

$$-X \sin \theta + Y \cos \theta = 0 \tag{6.51}$$

may be accomplished by the *vector servomechanism* shown at the top of Fig. 6.57*a*. Suppose that initially the servo shaft displacement is not equal to the correct value θ, so that Eq. (6.51) does not hold but

$$-X \sin \theta + Y \cos \theta = \epsilon$$

In the circuit of Fig. 6.57*a*, *a voltage equal to the "error" ϵ is computed and used to drive a servomechanism which will position the θ shaft so as to minimize the error ϵ.* The shaft displacement will, thus, be forced to assume the correct value θ. Function potentiometers on the θ shaft are then used to compute the voltage R according to the second equation (6.50). Potentiometers of the rectangular-card type will be found useful in the circuit of Fig. 6.57*a*.

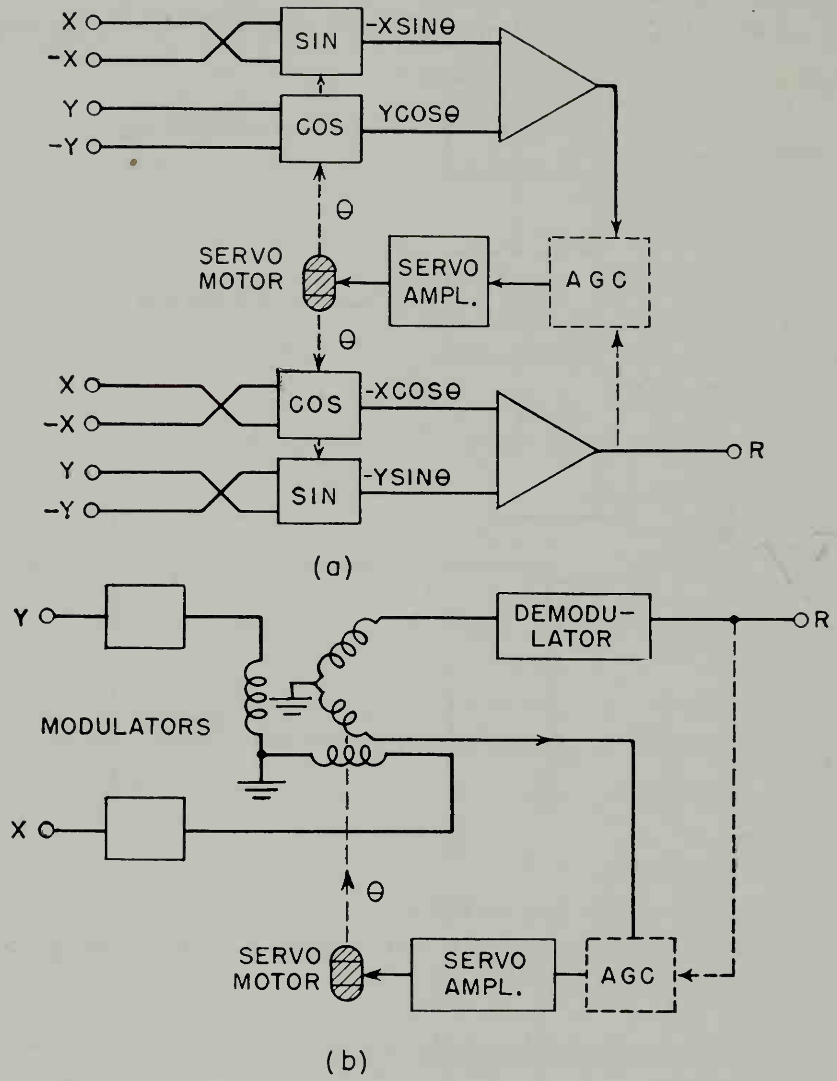

FIG. 6.57. Block diagrams of vector servomechanisms.

Figure 6.57*b* shows a similar arrangement in which the function potentiometers have been replaced by a single resolver driven by an a-c servo.

It may be seen that the loop gain of each vector servo shown in Fig. 6.57 is not constant but depends on the values of X and Y; in fact, the loop gain is proportional to the value of

$$R = \sqrt{X^2 + Y^2}$$

Since only servomechanisms with constant loop gain can be designed for optimal performance, it is expedient to include *automatic gain control*

(*AGC*) *circuits* (shown in dotted lines in Fig. 6.57) in the servo loop. Such circuits may consist simply of an amplifier stage using a variable-μ tube biased by a negative voltage whose value is proportional to R. Such a device will tend to keep the loop gain constant over some range of values of R; it is clear that no vector servo can operate if $X = Y = R = 0$.

A practical AGC unit used with a vector servo is discussed in the REAC Manual,[1] whereas the circuit of a vector servo without AGC is shown in Vol. 21 of the MIT Radiation Laboratory Series.[2] A circuit showing a vector servo with single-ended inputs is also shown in this same volume.[3] Accuracies attainable with vector servos are of the order of $\frac{1}{4}$ to 1 per cent in amplitude of the radius vector R and 0.5 deg in angle.

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[1] Reeves Instrument Corporation, New York City.

[2] Greenwood, Holdam, and MacRae, *op. cit.*, p. 479.

[3] *Ibid.*, 163.

CHAPTER 7

AUXILIARY CIRCUITS AND COMPUTER OPERATION

7.1. Introduction of Initial Conditions. HOLD Condition. *Introduction.* The necessity of introducing initial conditions of the machine variables into any differential analyzer was pointed out in Secs. 1.2 and 2.7. Differential equations are relations between *changes* of variables, and the initial conditions define the initial values from which these changes of variables must start. The number of initial conditions which must be given with a system of ordinary differential equations is equal to the *order* of the given set of differential equations, *i.e.*, the sum of the highest order derivatives in each differential equation of the set.

In computer circuits using capacitors to establish relations between changes of machine variables, the number of initial conditions to be introduced will, in general, be equal to the number of capacitors used. The initial conditions for the machine variables will be established if, for instance, each capacitor is given a correct *initial charge.*

If, as is most frequently the case, parallel-feedback integrators are used to relate changes of machine variables, an initial condition corresponding to each integrator must be established. This may be accomplished by two basic methods:

1. Each integrating capacitor may be charged initially, so that each integrator output voltage will start to vary from the correct initial value.
2. All integrating capacitors may be discharged before computation; a step-function voltage equal in value to the respective desired initial condition voltage must then be added to each integrator output voltage at the start of the computation.

The second method requires that the output voltages of all the integrators be forced to a reference level, usually zero, before computation, the initial-condition voltages being supplied as step functions equal in magnitude to the correct initial output voltages. Although this second method is somewhat easier to instrument, it can be used only with computers having recorders and computing elements capable of following a step function with sufficient accuracy. This requirement is satisfied by repetitive computers which are discussed in Sec. 8.5.

The first method is more applicable to "slow" d-c analog computers incorporating servomechanisms and recorders which cannot follow a step function accurately. With this first method, the output voltages of all computing elements are set to their initial values before computation. Accordingly, the gain control and reference level of the *recording devices* can be adjusted using the known initial condition voltages to calibrate the recorder scales.

Circuits for Setting up Initial Conditions. Before the start of each computer run, the output voltage X_o of each integrator is made to assume the correct initial value X_{o0} by initially charging the integrating capacitor. Since enforcement of the given differential equations by the computing elements would make it impossible to keep the variables set at their initial values, it is necessary to make the computer *inoperative* in the RESET or "initial" condition.

Figure 7.1*a* shows a possible arrangement for achieving these purposes. The relay K_R (one in each integrator) is energized in the RESET condition. In this condition, each integrator is disabled; the input grid terminal is grounded, and the amplifier output terminal is disconnected from the integrating capacitor. The computer is thus rendered inoperative, as required. The output terminal of the integrating capacitor is connected to the initial-condition voltage source shown. The desired initial-condition voltage is set by means of a potentiometer and will, then, charge the integrating capacitor so that the integrator output voltage assumes the correct initial value. Sufficient time must, of course, be allowed in each case to permit the capacitor to acquire its initial charge.

Note that a loading correction (see Sec. 4.1) must be applied to the reading of the initial-condition setting potentiometer.

With all initial conditions established in this manner, the computer run is initiated by means of a "compute" switch which deenergizes the relay K_R in each integrator. In the COMPUTE condition, the initial-condition setting circuit is removed from the integrator, which is now in its normal operating condition and connected to the other computing elements. The output voltage X_o will then vary from the preset initial value X_{o0} as prescribed by the given differential equations.

In another arrangement shown in Fig. 7.1*b*, the integrator is made inoperative by grounding the input terminals instead of the input grid. This circuit has certain advantages over that of Fig. 7.1*a* in that the impedance level at the amplifier grid is not radically changed at the start of the computation. In order to avoid the use of several relay contacts in summing integrators, one may also ground the amplifier grid through a resistance roughly equal to the resistance of the input resistors taken in parallel. In the circuit of Fig. 7.1*b*, the integrator output voltage will assume the desired initial-condition voltage only slowly, since the integrator input resistors are included in the capacitor charging circuit.

A third circuit for introducing initial conditions, illustrated in Fig. 7.2*a* and *b*, combines some of the advantages of the first two circuits and also avoids all switching at the amplifier output terminals. Here the control relay introduces changes in the integrating network and thus permits more rapid charging of the integrating capacitor. Figure 7.2*a* shows the complete circuit, whereas Fig. 7.2*b* shows the integrator in the RESET condition. The integrator has a new input resistor r, which is connected to the initial-condition source voltage E_o, and a new leakage resistor r_L. The output voltage X_o* of this circuit is given by Eq. (4.51); thus

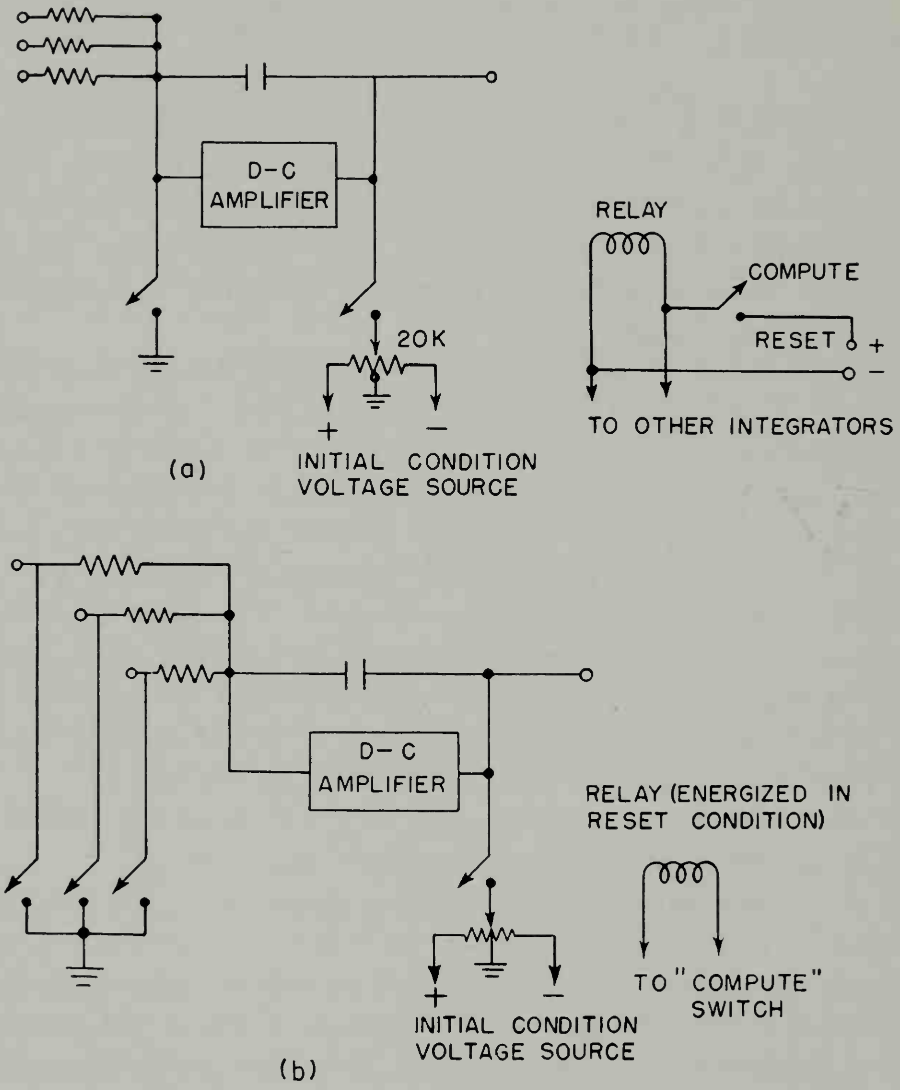
D-C AMPLIFIER
20K
+
−
INITIAL CONDITION VOLTAGE SOURCE
RELAY
COMPUTE
RESET
+
−
TO OTHER INTEGRATORS
(a)
D-C AMPLIFIER
RELAY (ENERGIZED IN RESET CONDITION)
+
−
INITIAL CONDITION VOLTAGE SOURCE
TO "COMPUTE" SWITCH
(b)

FIG. 7.1. Circuits for establishing initial conditions.

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* The reader should remember that X_o is given by Eq. (7.1) only in the RESET condition.

$$X_o = E_o \frac{A}{(1 - A)\frac{r}{r_L} + 1} \frac{1}{\dfrac{(1 - A)rC}{(1 - A)\frac{r}{r_L} + 1} P + 1} \tag{7.1}$$

Since E_o is a constant voltage, the reasoning of Sec. 4.5 shows that the integrator output voltage X_o will approach the value

$$X_{o0} = E_o \frac{A}{(1 - A)\frac{r}{r_L} + 1} \approx - E_o \frac{r_L}{r} \qquad |A| \gg 1, \left| A \frac{r}{r_L} \right| \gg 1 \tag{7.2}$$

the more rapidly the smaller the value of r_LC is; X_{o0} is the desired initial value of the integrator output voltage. The approximate form of Eq. (7.2) is usually sufficiently accurate. The reversal of sign in Eq. (7.2) must be remembered when actually making computer setups.

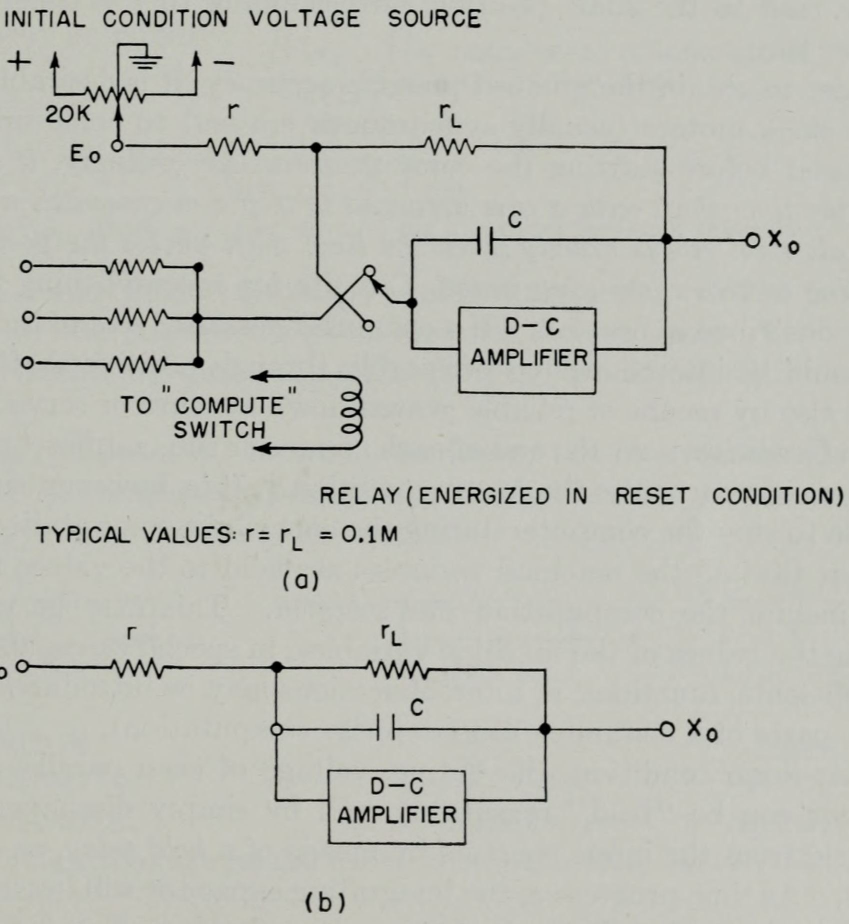
INITIAL CONDITION VOLTAGE SOURCE
+
−
20K
r
r_L
E0
C
X0
D-C AMPLIFIER
TO "COMPUTE" SWITCH
RELAY (ENERGIZED IN RESET CONDITION)
TYPICAL VALUES: r = r_L = 0.1M
(a)
r
r_L
E0
C
X0
D-C AMPLIFIER
(b)

FIG. 7.2. Circuits for establishing initial conditions.

In all the preceding circuits, the control relays may be arranged so that they are either energized or deenergized in the COMPUTE condition; the latter is preferable if the relays operate on alternating current. The value of the initial-condition voltage may be determined by means of a voltmeter. It is preferable, however, to use a constant-voltage source and a calibrated potentiometer and to refer all other voltage values in the computer to the voltage of this source.

Initial Settings of Computing Elements Other than Integrators. Summing amplifiers and phase inverters need not be made inoperative in the RESET condition. Their output voltages will automatically assume correct initial values determined by the initial values of the integrator output voltages. This is also true for computer servomechanisms whose shaft positions are determined by the values of their input voltages.

Time shafts, such as those used to drive function potentiometers, are usually driven by clock motors instead of servomechanisms. Such shafts must be *reset* to the shaft position corresponding to $\tau = 0$ before each computer run.

In order to obtain the greatest possible accuracy, it is desirable to permit the clock motors (usually synchronous motors) to come up to constant speed before starting the computation. Accordingly, *it is best to provide the time shaft with a cam arranged to trip a microswitch which will actuate all reset relays exactly when the time shaft passes the position corresponding to zero computing time.* Circuits for accomplishing this purpose are described in Sec. 7.2. If a computer possesses *several* time shafts, they should be interconnected preferably through positive shaft connections or else by means of reliable geared-down synchro or servo systems.

HOLD *Condition.* At the end of each computer run, a differential equation solver is returned to the RESET condition. It is, however, sometimes desirable to stop the computer during the course of a computation in such a manner that all the machine variables are held to the values they had at the instant the computation was stopped. This may be useful for checking the values of the machine variables; in special cases, also, different coefficients, functions, or interconnections may be introduced for subsequent parts of a computer run (stepwise computation).

In this HOLD condition, the output voltage of each parallel-feedback integrator can be "held" reasonably well by simply disconnecting the input grid from the input resistors by means of a *hold relay,* as shown in Fig. 7.3. As time progresses, the integrating capacitor will lose its charge at the initial rate $100/R_LC$ per cent per second, where C and R_L are the capacitance and the leakage resistance of the integrating capacitor, respectively. For integrating capacitors of good quality, this rate may be less than 0.002 per cent/sec. It is seen, however, that, unlike mechanical and electromechanical differential analyzers, *a d-c analog computer cannot be kept in the* HOLD *condition for an indefinite time.*

Summing amplifiers and servomechanisms will be kept in the HOLD condition by the integrator output voltages, but provisions must be made to stop all time motors or clock motors associated with the computer. If such motors have a tendency to coast when stopped, they must be provided with brakes effective in the HOLD condition.

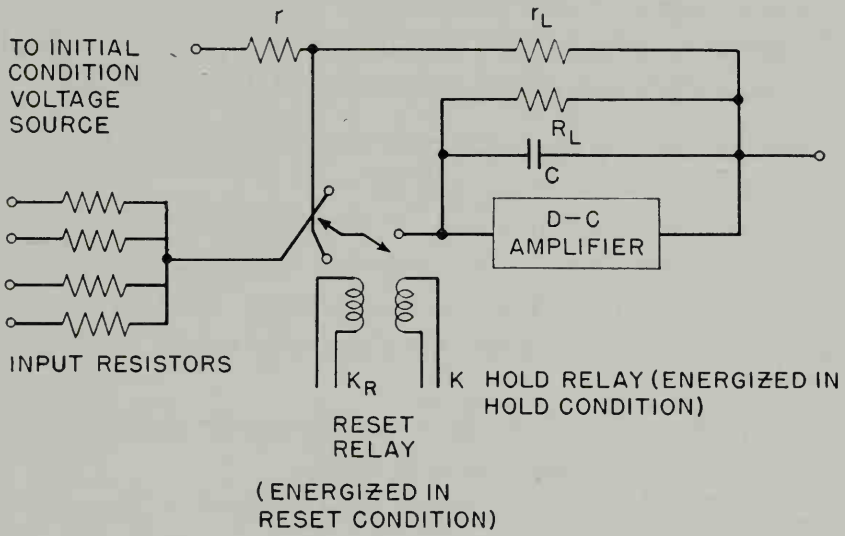
r
r_L
TO INITIAL CONDITION VOLTAGE SOURCE
R_L
C
D-C AMPLIFIER
INPUT RESISTORS
K_R
K HOLD RELAY (ENERGIZED IN HOLD CONDITION)
RESET RELAY
(ENERGIZED IN RESET CONDITION)

FIG. 7.3. Hold and reset circuits for a parallel-feedback integrator.

Figure 7.4 shows once again the configurations of a parallel feedback integrator in the COMPUTE, RESET, and HOLD conditions. The control circuits for actuating the reset and hold relays are described in detail in Sec. 7.2.

With the computer in the HOLD condition, the interrupted computer run may be continued by deenergizing the hold relays K; the machine will then return to the COMPUTE condition.

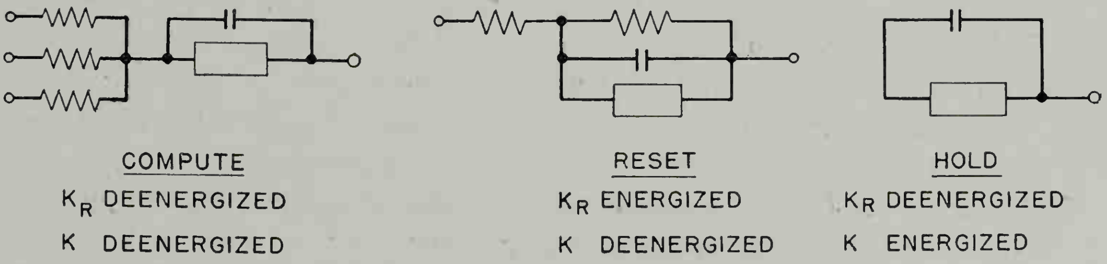

FIG. 7.4. Configuration of a parallel feedback integrator in the RESET, COMPUTE, and HOLD conditions.

7.2. Control Circuits. *Introduction.* The purpose of the control circuits is to initiate, stop, and supervise the computing operation; to indicate certain gross errors; and to protect the computer from overloads. The design of these circuits is a very important problem. The control circuit design may be considered as a connecting link which ties all the various computing elements and the associated equipment together into one integrated operating unit, a computing machine. The computer must be easy to use for the operating personnel. The control operation must be foolproof as well as practical and convenient from the "human engineering" viewpoint, for the control circuits must be used over and over and often by nontechnical personnel.

Control Functions. The *control and indicating devices* of a typical d-c analog differential equation solver will comprise the following:

1. *Power-supply controls* (switches and/or push buttons) for turning the computer power supplies on and off.
2. *Operating controls* (switches and/or push buttons) which permit the operator to start and stop the operation of computing elements and recording equipment. *The most important operating controls are those governing the action of the reset and hold relays on the integrators* (Sec. 7.1).

Note: The operating controls may be physically separated from the power-supply controls in order to avoid errors.

3. *Control relays and automatic timers* may be used to operate the controls automatically.
4. *Indicator lights* for indicating conditions such as

   SAFE
   STAND-BY
   RESET (READY)
   RECORDER ON
   COMPUTE
   HOLD
   MOTOR ON

   Such indicator lights help to supervise the control operations. Large bull's-eye-type indicator lights can be more easily seen from a remote-control position and are also a joy for visiting firemen. There is no strict convention governing the colors of the various indicator lights, but these colors should be chosen so that no possibility of confusing one light with another exists.
5. *Overload indicators* (lights, buzzers, etc.) warn the operator when an overload occurs anywhere in the computing circuits.
6. *Remote-control stations* enable the operator to control the computer from any location in the computing room.
7. *Dials* serve to indicate clock motor or servomechanism shaft displacements corresponding to the values of various machine variables.
8. *Control devices* of associated equipment.

A Simple Control System for a D-c Analog Computer. The physical operation of the controls just outlined can be demonstrated best by analyzing a basic control system for a simple differential equation solver. Such a control system is shown in Fig. 7.5. The control sequence for

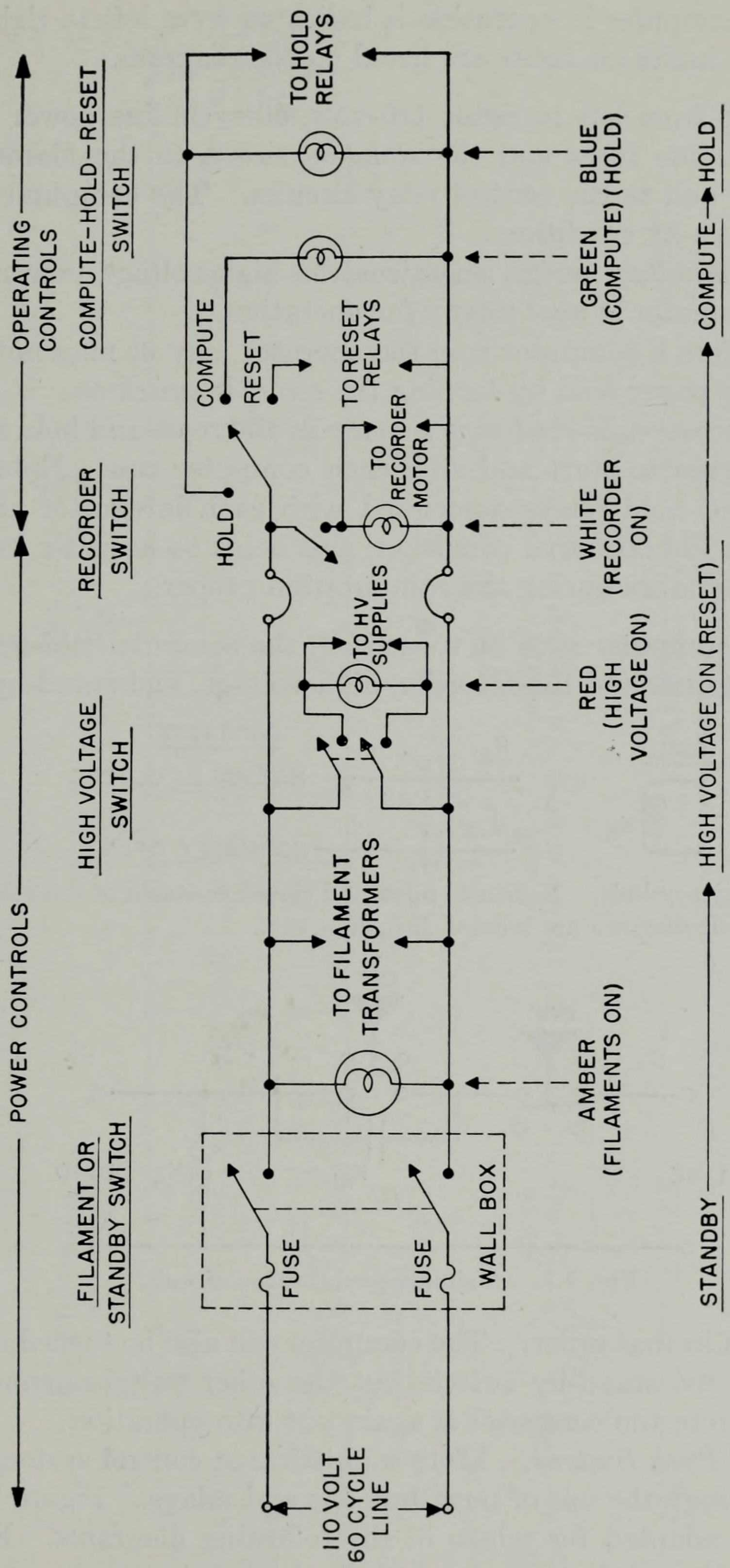

Fig. 7.5. Basic control system for a d-c analog computer.

placing the computer in operation is indicated from left to right, and all switches and indicator lights are listed on the diagram.

1. Reading from left to right, 110-volt, 60-cycle line power is applied through line fuses and the *stand-by switch* to the filament transformers and to the control relay circuits. The computer is now in the STAND-BY condition.
2. The *high-voltage switch* energizes the high-voltage power supplies. The computer is now READY for operation.
3. Just before a computer run, the operator may or may not start the recorder paper feed by turning the *recorder switch* on.
4. The *compute-hold-reset switch* controls the reset and hold relays and thus serves to start and stop each computer run. Note that the reset and hold relays associated with each integrator are deenergized in the COMPUTE condition; this helps to keep a-c hum out of the integrators during the computation proper.

When the computer is to be turned off, the compute-hold-reset switch is turned to RESET and the recorder, high-voltage, and stand-by switches are turned off in that order. The computer can also be turned off at once by means of the stand-by switch, but the other switches should all be turned off before the computer is again put into operation.

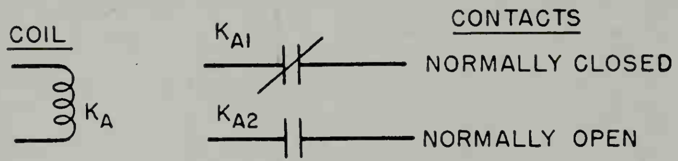

FIG. 7.6. Relay symbols. Normally open and closed contacts of the relay K_A anywhere in a circuit diagram are labeled K_{A1}, K_{A2}, etc.

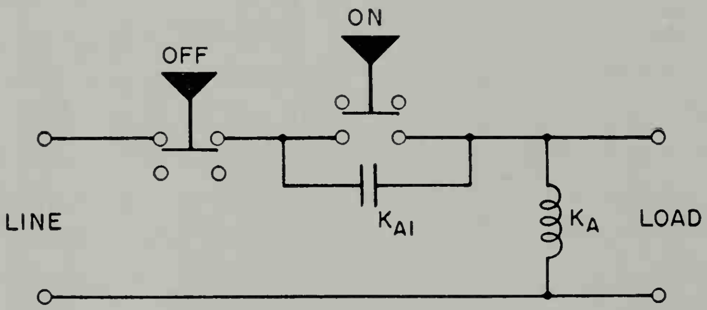

FIG. 7.7. A simple push-button circuit.

Relays and Push Buttons. More sophisticated control systems may be obtained through the use of push buttons and relays. Figure 7.6 shows the notation adapted for relays in the following diagrams. Figure 7.7 shows a simple push-button system. Each push button is a two-circuit switch. The load is energized by touching the ON button; this also

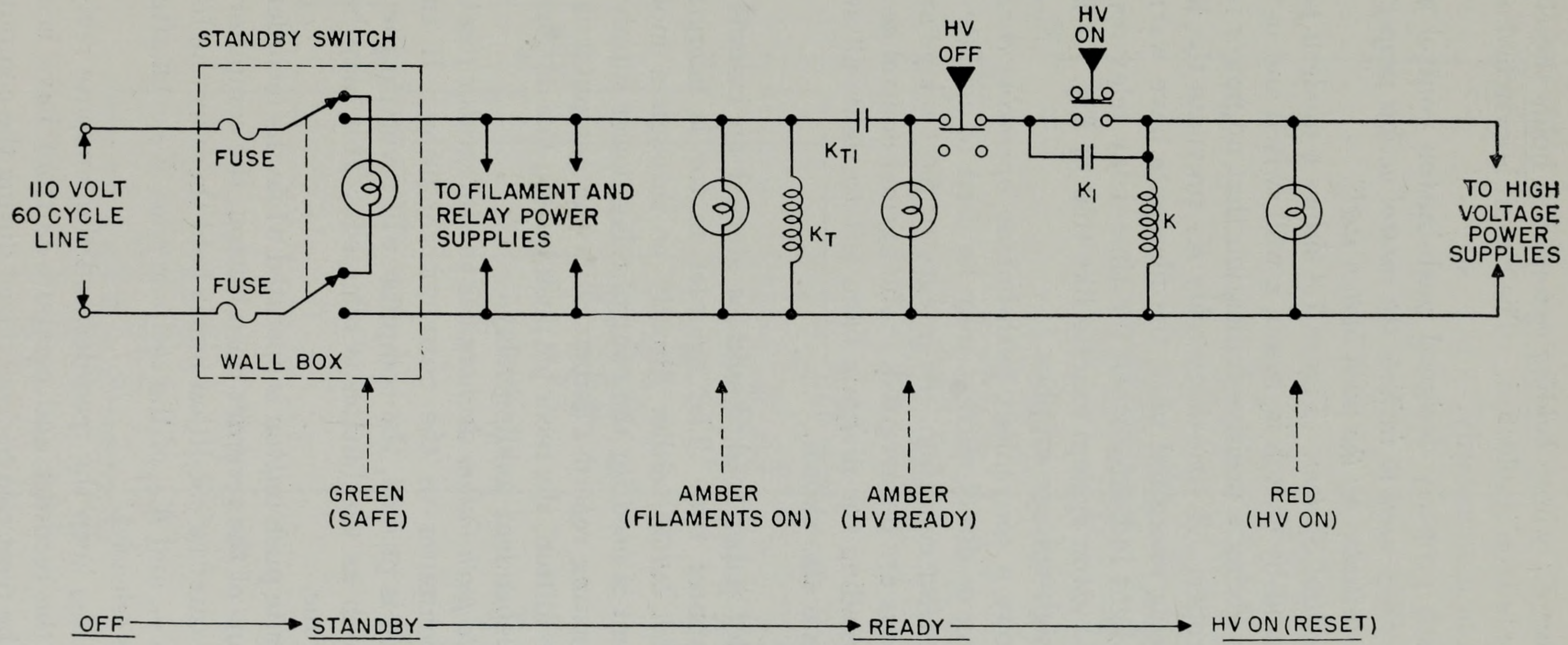

FIG. 7.8. A push-button power supply control system.

actuates the relay K_A whose *holding contact* K_{A1} holds the circuit closed even after the ON button is released. The circuit is switched off by touching the OFF button momentarily.

Note that, with properly designed push-button control systems, *the holding relay is rarely used to initiate the control action proper; the control action is usually initiated by the push button itself.*

Push-button Control System. Figure 7.8 shows a modern *power control* system. The stand-by switch is, again, a wall switch and incorporates a green *safety light* giving a positive indication that no power is applied to the computer proper. *A time-delay relay* K_T prevents the high-voltage supplies from being energized until the filaments have warmed up; an amber "ready" light indicates when the time-delay relay contact K_{T1} is closed. A push-button system exactly like that of Fig. 7.7 is then used to energize the high-voltage supplies.

Figure 7.9 shows a convenient push-button *operating control* system. Either alternating or direct current may be used to actuate the control relays. In the RESET condition, the integrator reset relays are energized and the hold relays are deenergized. The normal control sequence may be followed by reading the diagram from left to right; all switches and indicator lights are also shown.

1. The recorder paper feed should be started an instant before the actual computer run. The recorder motor is energized by the *recorder push button*, either directly or through a motor relay if direct current is used for the control relay power supply. In order to avoid wasting recorder paper while the computer is not in the COMPUTE condition, the recorder push button has not been provided with an independent holding relay.
2. The *compute push button* deenergizes the integrator reset relays and puts the computer in the COMPUTE condition. If the compute button alone is pushed, the computer will revert to the RESET condition as soon as the button is released. This may be useful for short test runs.
3. If the compute push button is operated while the recorder is made to run by means of the recorder push button, the computer is not only put in the COMPUTE condition but is held there by the four contacts K_{C1}, K_{C2}, K_{C3}, and K_{C4} of the *compute relay* K_C even after the pushbuttons are released.

   K_{C1} and K_{C2} keep the recorder and the compute relay energized even after the recorder and compute buttons have been released. K_{C3} keeps the reset relays deenergized during the computer run, and K_{C4} permits the hold circuits to be energized.

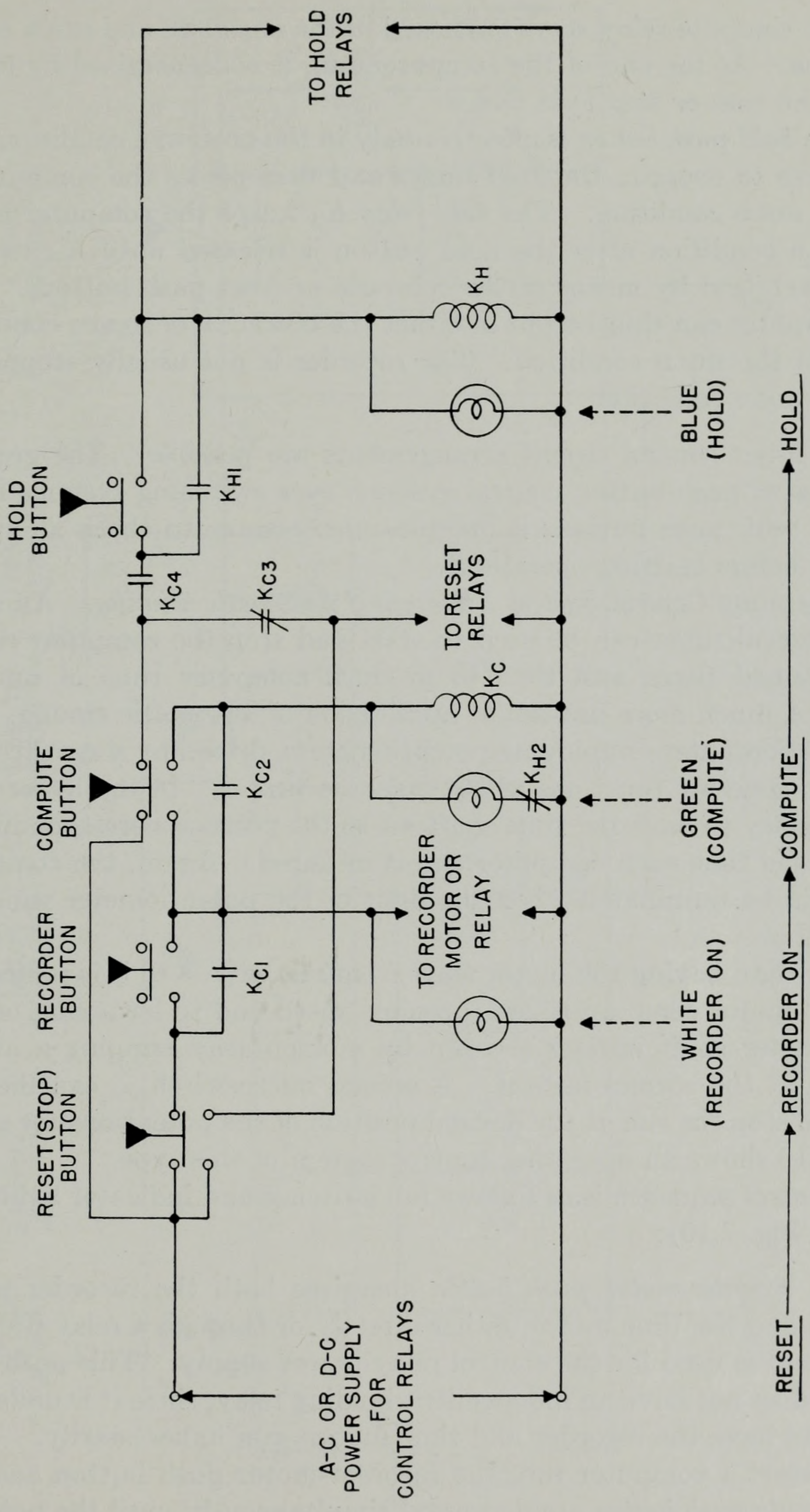

Fig. 7.9. A push-button operating control system.

4. The compute relay stays energized in the COMPUTE and HOLD conditions. At the end of the computer run, it is deenergized by means of the *reset or stop push button.*
5. The *hold push button* is effective only in the COMPUTE condition. It serves to energize the hold relays and thus places the computer in the HOLD condition. The *hold relay* K_H keeps the computer in the HOLD condition after the hold button is released until K_H itself is deenergized by means of the compute or reset push buttons. The computer can thus be put in either the COMPUTE or RESET condition from the HOLD condition. The recorder is not usually stopped in the HOLD condition.

Many other control circuit arrangements are possible. The greatest advantage of push-button control systems over switching systems is the fact that with push buttons it becomes unnecessary to check all switch positions before starting operations.

An Operating Control System Employing Automatic Timing. An automatic interval timer can be used to start and stop the computer run at predetermined times and thus to produce computer runs of uniform length. A much more important application of automatic timing, however, is in computers employing potentiometers, driven by a synchronous motor, to generate functions of the machine time τ. In the latter case, it is necessary to have the time shaft set to the position corresponding to $\tau = 0$ at the time each computer run is initiated. Again, the computer run should be terminated when the limit of the potentiometer windings is reached.

Rather than having the motor start from rest at $\tau = 0$, it is preferable to let the motor come up to synchronous speed and to let a cam on the potentiometer shaft initiate the run by momentarily tripping a microswitch μ_1 at the correct instant. A second microswitch μ_2 can then be tripped to stop the run at the desired position of the potentiometer shaft. Figure 7.10 shows an operating control system of this type.

The control sequence is as follows (all switches and indicator lights are shown in Fig. 7.10):

1. The *recorder-motor push button* energizes both the recorder paper feed and the time motor, either directly or through a relay if direct current is used for the control relay power supply. This push button does not have an independent holding relay, since it is undesirable to have the recorder and time motors run unnecessarily.
2. To start a computer run, the recorder-motor push button *and* the *compute push button* are depressed simultaneously until the potentiometer shaft reaches the position corresponding to $\tau = 0$. The cam

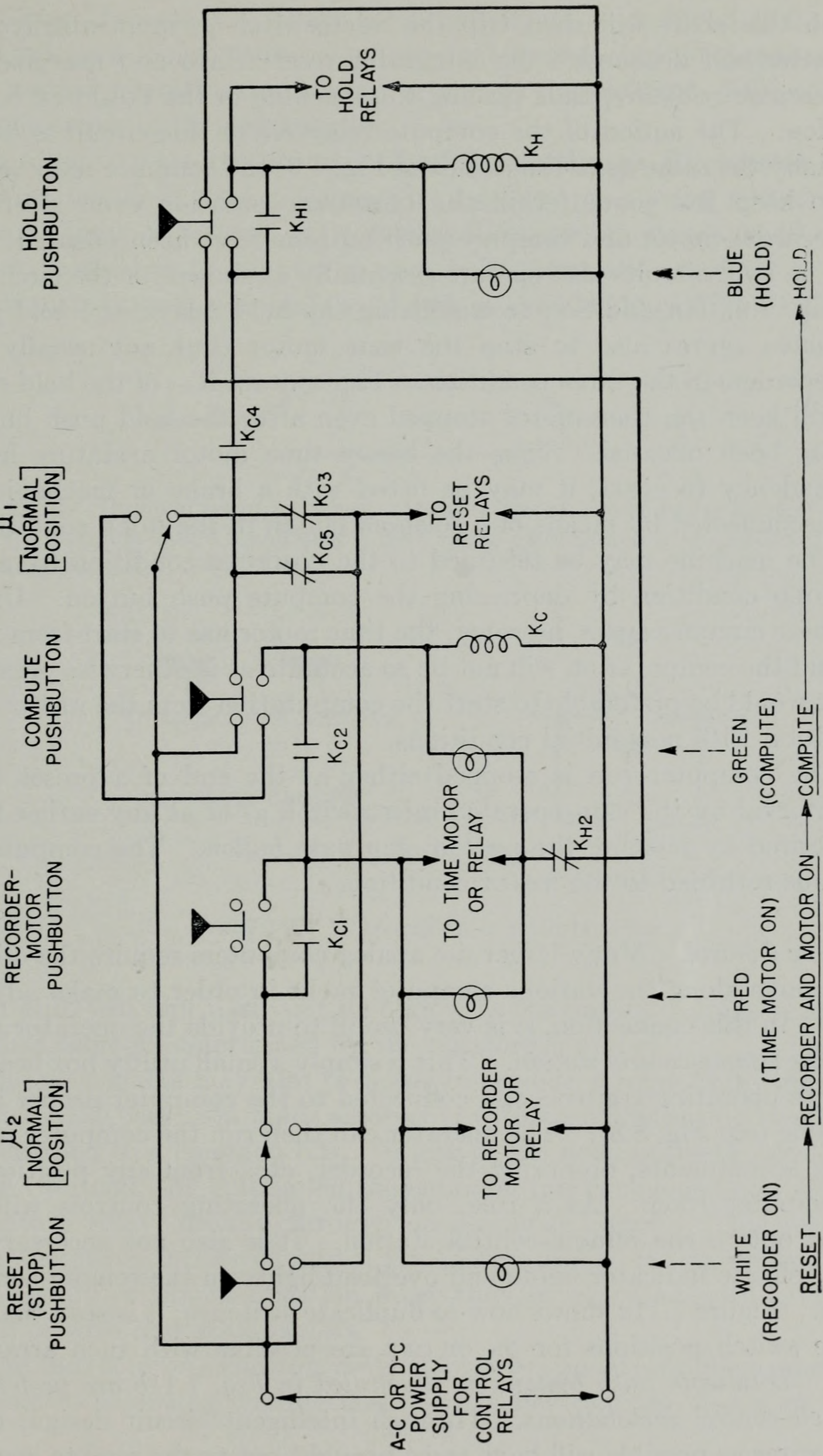

FIG. 7.10. Push-button control system with automatic timing.

on this shaft will then trip the microswitch μ_1 momentarily; the latter will deenergize the integrator reset relays and energize the *compute relay* K_C, thus placing the machine in the COMPUTE condition. The action of the compute relay K_C in this circuit is essentially the same as in the circuit of Fig. 7.9; the compute relay serves to keep the computer in the COMPUTE condition even after the recorder-motor and compute push buttons have been released.

3. The hold circuits also operate essentially as shown for the circuit of Fig. 7.9. In addition to energizing the hold relays, the *hold push button* serves also to stop the time motor (but not usually the recorder) in the HOLD condition. The contact K_{H2} of the hold relay will keep the time motor stopped even after the hold push button has been released. Since the heavy time motor armature has a tendency to coast, it may be fitted with a brake or mechanically disconnected by means of a solenoid clutch in the HOLD condition.
4. The machine may be returned to the COMPUTE condition from the HOLD condition by depressing the compute push button. Under these circumstances, however, the time motor has to start from rest, and the computation will not be so accurate as is otherwise possible. It would be preferable to start the computation from the RESET condition with new initial conditions.
5. The computer run is stopped either at the end of a preset time interval by the cam-operated microswitch μ_2 or at any earlier time desired by pushing the *reset or stop push button*. The computer is thus returned to the RESET condition.

Remote Control. Many larger d-c analog computers require the operator to move along the various computer racks in order to make adjustments. In this connection, it is very useful to provide the operator with a *portable remote-control station.* This is simply a small utility box bearing duplicate operating controls and connected to the computer proper by a long cable (see Fig. 8.8). The operator can then run the computer while making adjustments, observing the recorder, etc., from any position in the computer room. As a rule, only the operating controls will be brought out to the remote-control station. It is also not necessary to have duplicate indicator lights and overload lights on the remote-control station. Figure 7.11*a* shows how to duplicate switches; it is seen that no definite switch positions for ON or OFF are possible with such arrangements. *Duplicate push buttons as illustrated in Fig.* 7.11*b are preferable in remote-control installations.* Through intelligent circuit design, only as few wires as possible will have to be brought out to the remote-control station in each case.

Overload Indicators. Every computer should be designed so that overload voltages do not result in any physical damage, but voltages exceeding the normal operating ranges of the computing elements will still cause serious errors in the computation.

In order to warn the operator of overload voltages, the current flowing in a limiter diode (Sec. 6.7) may be used to trip a relay and actuate a warning light or buzzer. For overload voltages in excess of 70 volts, a better method is to connect a *neon bulb* across the amplifier or servomechanism output terminals as shown in the circuit of Fig. 7.12*a*. The neon bulb will first flash and then glow as the output voltage reaches an *absolute value* predetermined by the resistors r_1 and r_2.

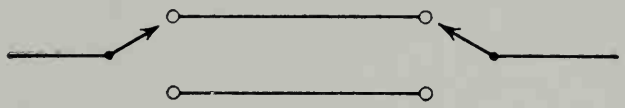

FIG. 7.11*a*. Duplication of switches. Definite switch positions for ON and OFF are lost; the circuit is closed if and only if both single-pole–double-throw switches contact the same channel.

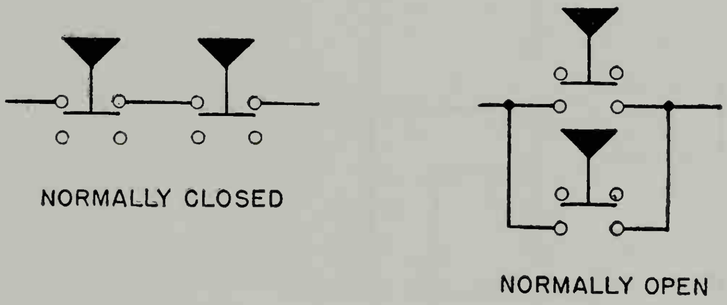

FIG. 7.11*b*. Duplication of push buttons.

The neon bulbs may also be made to actuate a relay.[1] In the circuit of Fig. 7.12*b*, the neon bulbs associated with each computing element are all returned to a common bus. If any of the neon bulbs should fire due to an overload, the voltage on the common bus is changed and serves to actuate a relay, usually through an amplifier circuit. This relay can then energize a buzzer or return the computer into the RESET condition. The computer run in question must then be repeated with a changed setup or with new scale factors.

The *special error-indicating circuits* for operational amplifiers employing automatic balancing circuits described in Sec. 5.6 are also a species of overload-indicating devices. The error-indicating neon bulbs may be returned to a common bus in the manner described for overload lights.[1]

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[1] Garwin, *Rev. Sci. Instruments*, June, 1950.

7.3. Recording Equipment. *Introduction: Recorder Scales.* The recording equipment associated with d-c analog computers automatically

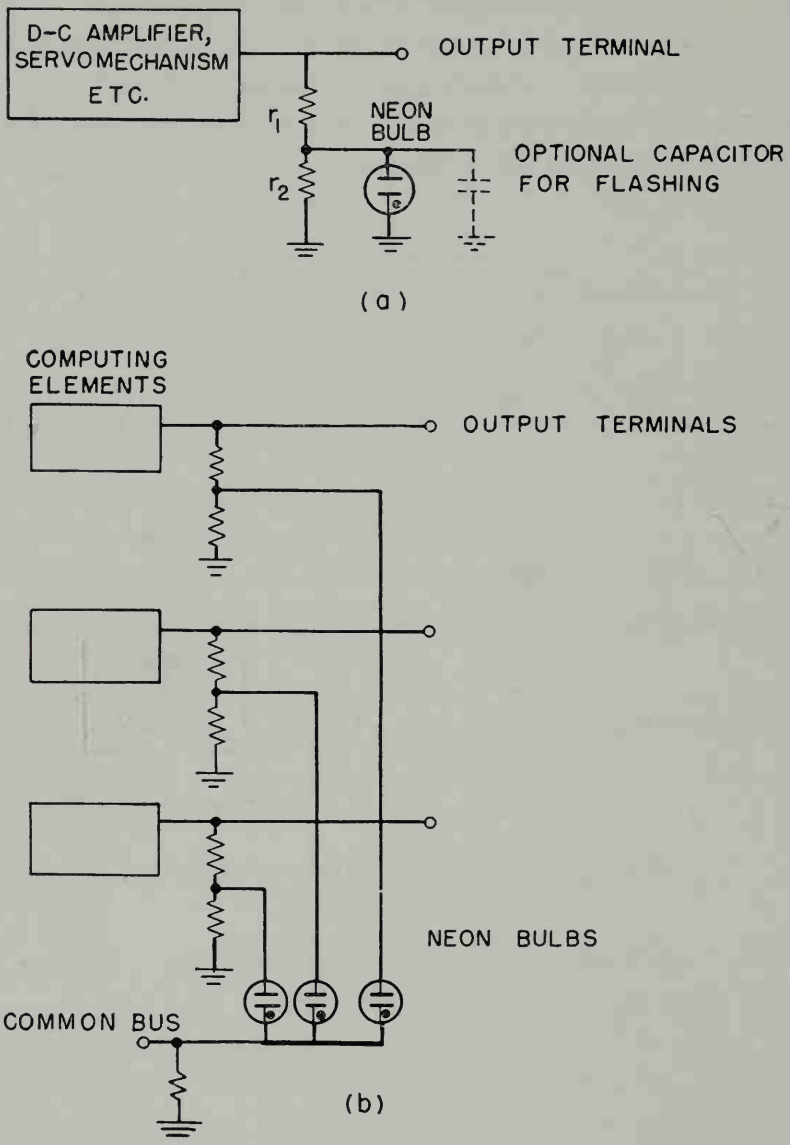

FIG. 7.12. Neon-bulb overload indicators.

plots graphs of one or more voltages (dependent machine variables) against the time τ as the independent variable. Some types of recorders also permit the plotting of one voltage (dependent variable) against another voltage or dependent machine variable.

All instruments used to plot d-c analog-computer solutions are, essentially, *recording voltmeters* whose deflections are proportional to variations of the machine variables. As a rule, the records are made on coordinate paper bearing scale divisions to indicate the values of voltage and time recorded. In some recorders, the number of scale divisions per machine unit or *recorder sensitivity* K is fixed for each of several *ranges* selected by means of a *range switch*. Figure 7.13a shows how different values of K may be obtained by means of a simple attenuator preceding the recording device. In other types, the *recorder scale factor* A_x for a problem variable x, or

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[1] See W. F. Gunning and R. S. Mengel, unpublished communication, RAND Corporation, Santa Monica, Calif., for a complete circuit.

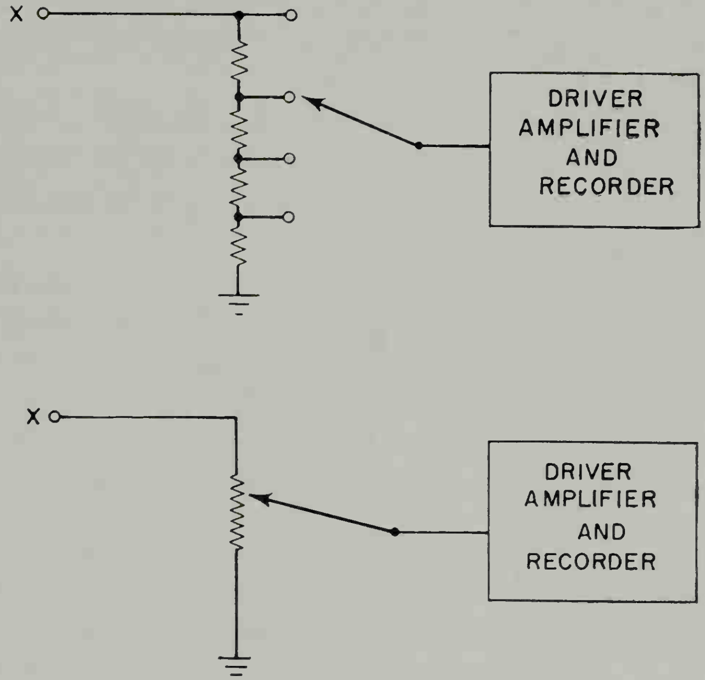

FIG. 7.13. Two methods for adjusting recorder sensitivity.

$$\left.\begin{aligned} A_x &= \frac{\text{recorder scale divisions}}{\text{unit problem variable}} \\ &= \frac{\text{recorder scale divisions}}{\text{machine unit}} \; \frac{\text{machine unit}}{\text{unit problem variable}} \\ &= Ka_x \end{aligned}\right\} \tag{7.3}$$

[where a_x is the scale factor proper (see Sec. 2.2) relating the problem variable x to the corresponding machine variable X], is continuously adjustable by varying the recorder sensitivity K. With the latter arrangement, it is particularly easy to make the recorder scale divisions correspond to convenient values of the problem variables. The records will then be more useful and easier to read. Recorders of this type may have to be calibrated for each new problem, *e.g.*, by means of a potentiometer as shown in Fig. 7.13b. This recorder *sensitivity control* should be equipped with a positive shaft lock lest the setting be disturbed accidentally after calibration.

The following is a list of properties desirable in recording devices used for computing purposes.

1. *Accuracy of calibration; frequency response.* The recorders should have an accuracy comparable to that of the computer in question (0.1 to 3 per cent) and should maintain this accuracy under all conditions of signal amplitude and frequency occurring during computation.
2. *Direct-recording feature.* The recorders should be capable of producing a record *immediately available* without processing, since computer records are frequently needed at once in order to determine the further course of the computation.
3. Records should be reasonably *permanent* and capable of being *reproduced* by standard photographic printing processes. Semitransparent recording paper is useful in this latter respect.
4. *Multichannel operation* makes it possible to record several machine variables on the same time scale. Records of this type are often particularly convenient for purposes of interpretation and comparison.

Survey of Recording Devices. The choice of a recorder for a given computer application depends chiefly on the accuracy required, on the frequency response demanded by the problem conditions, and on the time scale used. In the following paragraphs the most frequently used types of recording devices are listed together with some of their pertinent properties.

Cathode-ray Oscillographs. The frequency response of the familiar cathode-ray oscillograph exceeds that of all presently used computing elements.

The accuracy of cathode-ray oscillographs is low (5 to 10 per cent) if external scales are used for reading the recorded values. The accuracy may be increased to a remarkable extent (about 0.5 per cent) by introducing self-calibrating features, such as maker pips and reference lines, into the display.

Whereas a cathode-ray oscillograph does not permit direct recording, it may be possible to watch the display before or even while it is being photographed. Cameras with self-contained rapid film-processing devices (Polaroid Land camera) make it possible to obtain the finished oscillograph record within 2 min after photographing the oscillograph screen.

Other Electromechanical Oscillographs. A number of devices permit photographic recording by means of a light beam deflected by a mirror positioned through some current- or voltage-sensitive device (moving coil, string, crystal, etc., types of galvanometer). Such recorders are widely used for various commercial purposes. Some of these devices are accurate to better than 2 per cent and may have a frequency response flat up to 15,000 cps because of the low inertia of their moving parts. The inconvenience of photographic recording, however, constitutes a serious obstacle to the use of such recorders in many computer applications.

Direct-writing Magnetic Oscillographs. Most of the computer records shown in this book were obtained on direct-writing magnetic recorders of the general type shown in Fig. 7.14. Such devices consist of a paper feed mechanism and as many *pen motors* as there are recording channels. Each pen motor is essentially a d'Arsonval meter movement with a pointer bearing a lightweight writing device at its end. In the six-channel direct-inking recorder shown in Fig. 7.14, the meter pointers are hollow and feed ink onto the recording paper at the point of contact. In the perhaps more desirable arrangement shown in Fig. 7.15, the direct-inking pens are replaced by tiny heated wires which produce black traces on special heat-sensitive paper. Direct-inking magnetic oscillographs are capable of fair accuracy (2 to 5 per cent). The frequency response of some types may be made flat up to 100 cps through partial electrical compensation of the pen inertia. These devices may be made sufficiently small for convenient multichannel operation.

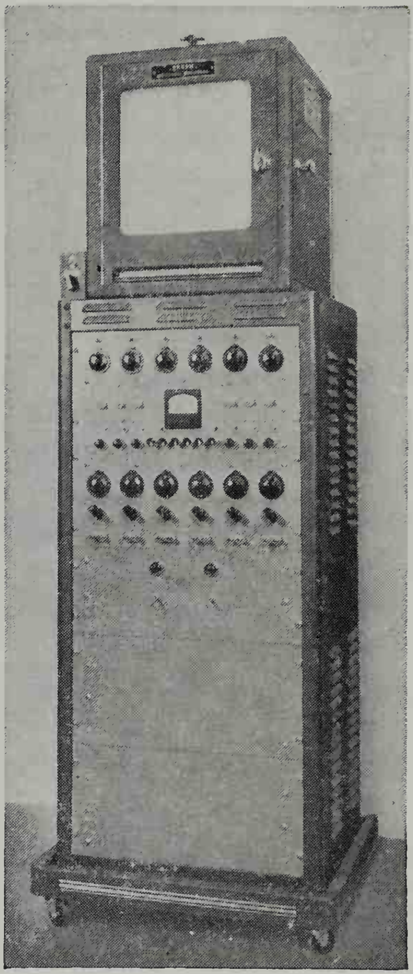
FIG. 7.14. Six-channel, direct-inking magnetic oscillograph (Brush) with Goodyear driver-amplifier assembly.

Recorder Driver Amplifiers. Magnetic oscillographs require currents of the order of 25 ma for full-scale deflection. Since such currents cannot be supplied from most computer amplifiers, current amplifiers (low-impedance drivers) are needed to drive the pen motors. Such driver amplifiers complete with power supplies may be purchased together with the recorders, but the prices of many commercial models are so very high that it may well pay to construct a set of amplifiers in the laboratory and to operate them from the main computer power supplies.

The principles of d-c amplifier design given in Chap. 5 apply to the design of recorder drivers, so that it will suffice here to present a number of typical circuits.

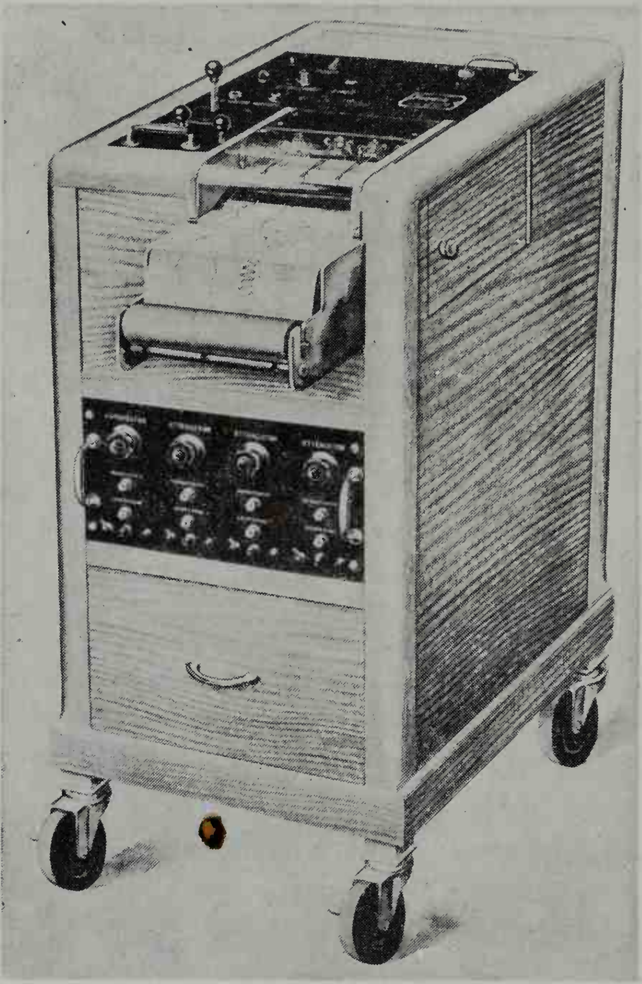

FIG. 7.15. Four-channel magnetic oscillograph for direct recording on heat-sensitive paper (Sanborn), used with the REAC computer of the Reeves Instrument Corporation, New York.

The *differential cathode-follower* circuit shown in Fig. 7.16 has better linearity than a simple cathode follower; besides, the bridge-balanced arrangement minimizes the unbalancing effects of all supply voltage variations. The current gain of the circuit is

$$\frac{\text{Load current}}{\text{Input voltage}} = \frac{g_m}{2 + (\mu + 1)\dfrac{R_L}{r_p} + \dfrac{R_L}{R_k}} \tag{7.4}$$

where g_m and r_p refer to one of the two identical tubes. The operating point must be determined from the tube characteristics (see Sec. 5.2). The linearity of the circuit may be improved at the expense of the current gain by introducing a resistance in series with the load. Typical circuit values are shown in Fig. 7.16.

Figure 7.17 shows a differential amplifier with preamplification. Further refinements may include the use of limiters (Sec. 6.7) to prevent overloading the pen motor; the latter is endangered particularly if one of the differential cathode-follower tubes should burn out.

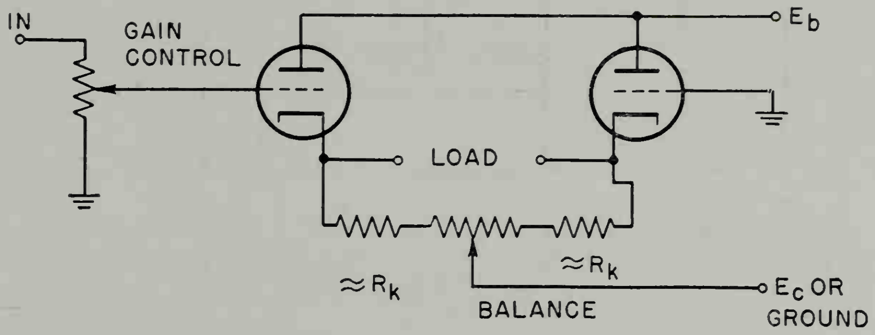

FIG. 7.16. Differential cathode follower. The current output may be increased if larger tubes and lower values of R_k are used. If several tubes are used in parallel, 1,000-ohm resistors should be inserted in series with each grid to prevent parasitic oscillations.

Typical Design Values

Two 6AS7G (two sections each side), R_k = 1,000 ohms, 50 watts
Four 6SN7 (four sections each side), R_k = 6,000–10,000 ohms, 20 watts

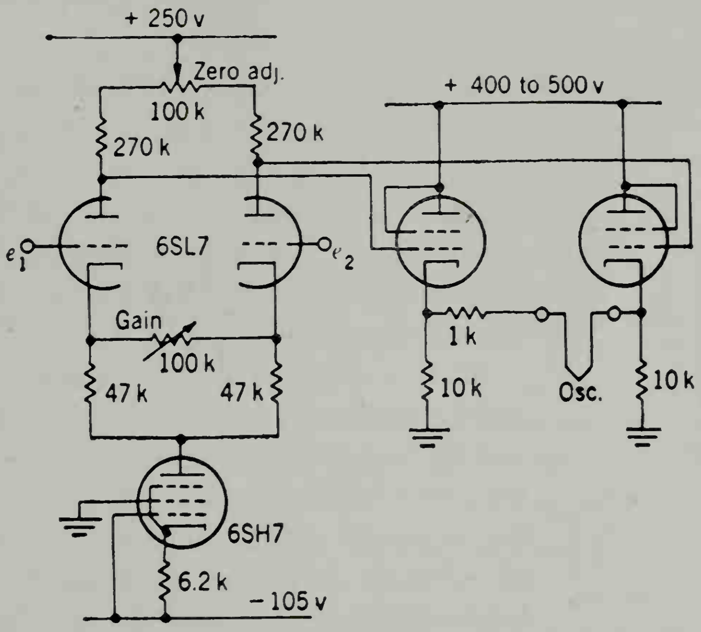

FIG. 7.17. Differential cathode follower with preamplification (G. E. Valley and H. Wallman, *Vacuum Tube Amplifiers*, M. I. T. Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948).

Figure 7.18 shows a more sophisticated low-impedance driver possessing high linearity and constant gain. This is essentially a high-gain d-c amplifier with cathode-follower output and current feedback; the output tube itself limits the output current in both directions through cutoff and grid current, respectively.

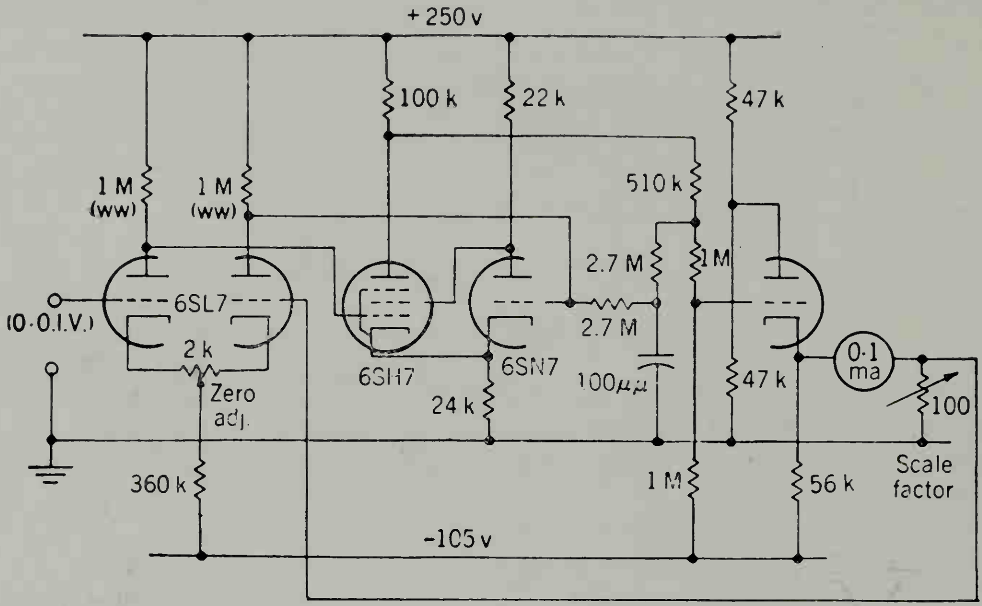

Fig. 7.18. A high-quality recorder driver amplifier (G. E. Valley and H. Wallman, *op. cit.*).

Figure 7.19 shows a simple equalizing network which may be inserted ahead of a driver amplifier to compensate in part for the poor high-frequency response of recorder pens. By means of such networks or equivalent feedback arrangements,[1] it is possible to keep the accuracy of certain magnetic recorders within 3 per cent at frequencies up to 100 cps.

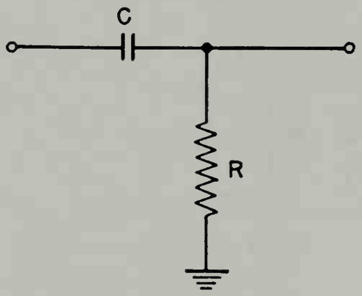

Fig. 7.19. Simple equalizing network inserted ahead of driver amplifier to improve the high-frequency response of a magnetic recorder.

Servo Tables. Voltage-actuated servomechanisms similar to the ones used to position computing potentiometers (see Sec. 6.4) may be used as reliable and accurate means for positioning recorder pens. Accuracies of 0.1 per cent of full-scale deflection are possible. The paper feed may be accomplished by a clock motor or by a second servomechanism; the latter arrangement makes it possible to plot one voltage with respect to another voltage. The cost of servo recorders need not be excessively high. The most serious limitation of servo recording devices is their poor high-frequency response; few if any are useful at signal frequencies above 5 cps.

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[1] Christian, D. R., *AIEE Paper* 47-226, presented at the AIEE meeting in Dayton, Ohio, Sept. 23–25, 1947. See also H. B. Shaper, Electrodynamic Direct-Inking Pen, *Electronics*, March, 1946.

Figure 7.20 shows a servo recorder developed by the Reeves Instrument Corporation. One servomechanism positions the pen parallel to the axis of a cylindrical drum carrying the paper. The paper feed is accomplished by rotating the drum through a second servomechanism. This device permits a recording accuracy of about 0.2 per cent at signal frequencies below 2 cps. This particular recorder is arranged so that it may also be used as a *curve follower* (see Sec. 6.5).

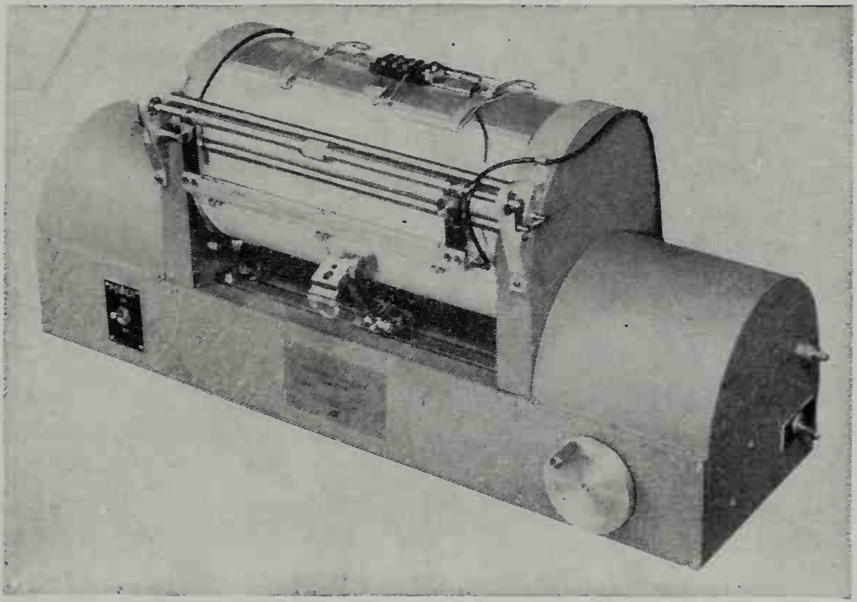

FIG. 7.20. Servo recorder developed by the Reeves Instrument Corporation, New York. This servo-driven drum also serves as a manually-driven input table and as a universal function generating potentiometer of the type shown schematically in Fig. 6.30 (see Sec. 6.5).

Figure 7.21 shows a larger servo plotting board or servo table. The pen carriage is positioned in two mutually perpendicular directions by two servomechanisms, so that no paper feed is necessary. It is also possible to construct servo tables which make plots in polar coordinates instead of rectangular Cartesian coordinates. Another interesting recording device is the so-called "crab" consisting of a recording pen traveling over the recording paper on a three-wheeled carriage. The crab is positioned by two servomechanisms. One of the latter determines the course angle by "steering" the carriage wheels, while the other one determines the distance traveled. Crabs are particularly useful for trajectory plotting. They are also convenient to use, since they will "walk" on any level plotting table and no special provisions for pen mounting or paper feed are required.

Other Types of Recorders. Facsimile-type or sweep-balance recorders[1] may permit direct recording on various types of paper at frequencies up to 20 cps. To the authors' knowledge, such recorders have not been used in computer applications, and space does not permit a discussion of them. It does not appear impossible to improve the high-frequency response of these devices sufficiently to make them suitable for computer applications, and this possibility might deserve future attention.

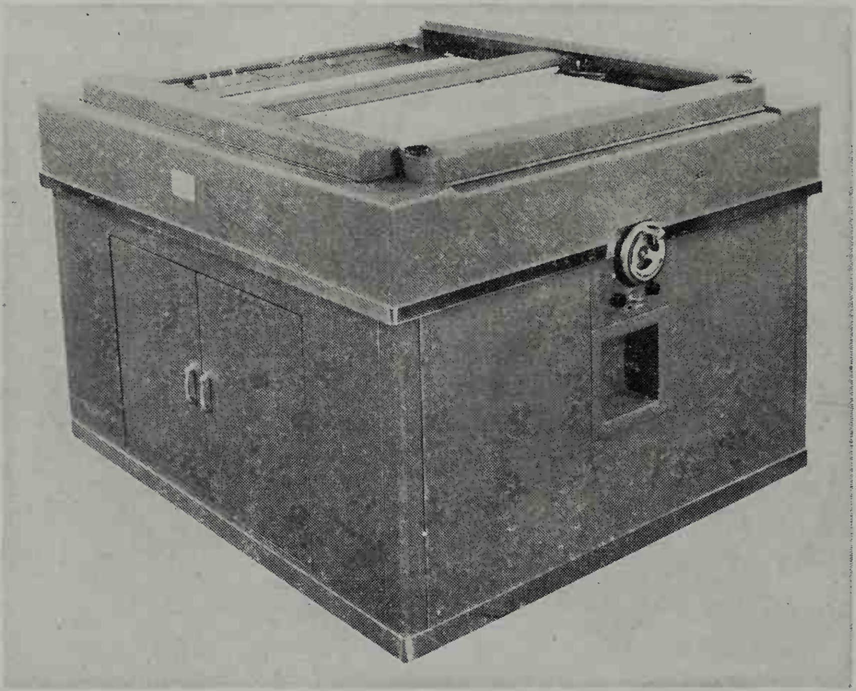

FIG. 7.21. Large servo plotting board developed by the Reeves Instrument Corporation, New York.

Dependence of Recorder Time Scales on Paper Speed. Since synchronous motors are used to drive the paper of most recording oscillographs, the paper speed will depend on the supply frequency. While the average frequency of a-c power lines is held constant within very small limits, instantaneous frequency deviations as large as 0.7 per cent are possible. Accordingly, one cannot rely on the power-line frequency for the recorder time scales in high-accuracy computers. This difficulty may be overcome in the following ways:

1. A record of the power-line frequency may be maintained and used to correct the recorder time scales.
2. A stable audio-frequency oscillator (tuning fork oscillator) may be used to generate *timing marks* on the recording paper.

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[1] See, for instance, Keinath, *Instruments*, **17**: 200, 1946.

3. The output of a tuning-fork oscillator may be amplified and fed to the synchronous motors used to drive function generators in the computer (see Sec. 6.5) as well as to the paper feed motor.

7.4. Computer Power Supplies. *Introduction.* Every electronic analog computer requires power supplies to supply the voltages needed for the operation of the computing elements and associated components. The power supplies necessary for the operation of a typical d-c analog computer are listed below:

1. *Filament power supplies* (1 to 24 volts alternating or direct current).
    a. Unregulated alternating current is suitable for filament operation in power output tubes and in the high-level stages of voltage amplifiers.
    b. Regulated alternating current or preferably regulated direct current should be used to supply the filaments of amplifier input tubes in order to minimize the drift due to cathode emission changes (see Sec. 5.1).
2. *Regulated high-voltage power supplies.*
    a. Positive plate supplies.
    b. Negative bias supplies.
    c. Positive and negative supplies for computing voltages.
3. *Power supplies for servomechanisms* are usually self-contained in each servo amplifier (see Sec. 6.4).
4. *Low-voltage d-c power supplies* for various control relays, solenoids, clutches, motors, and other associated equipment.

The designer must specify the currents and voltages required from each of these power supplies as well as the accuracy with which the voltage must be regulated.

Primary Power Sources. The operating voltages for most d-c analog computers are derived from the 110-volt, 60-cycle a-c line by means of *transformers and/or rectifiers.* In very large installations as well as in certain mobile applications, *motor-generator sets* are used to supply high-voltage direct current. Especially in smaller installations, storage and dry *batteries* should not be disregarded as sources of computer operating power; the use of batteries frequently makes electronic regulation unnecessary.

Filament Power Supplies. Unregulated alternating current (6.3 volts) is generally used to supply the filaments of high-level stages in the equipment, and this type of filament operation is well known. In order to avoid hum resulting from filament emission, it is advisable to return the center tap of the filament transformers to a point which is positive with respect to the cathodes in question.

The filaments of low-level stages of d-c amplifiers are frequently supplied with regulated alternating or direct current; the latter is somewhat preferable because the use of direct current helps to minimize hum pickup. The regulators used with either a-c or d-c filament supplies usually employ *saturable reactors* which tend to change their impedance as the result of current changes. Such regulators are generally available in the form of complete commercial units and are, thus, not usually built in the laboratory.

High-voltage Power Supplies. High-voltage supplies include positive plate voltage supplies, negative bias voltage supplies, and positive and negative supplies for computing voltages. Practically identical circuits are suitable for all these supplies; the output voltage will be positive or negative with respect to ground depending on which one of the output terminals is grounded.

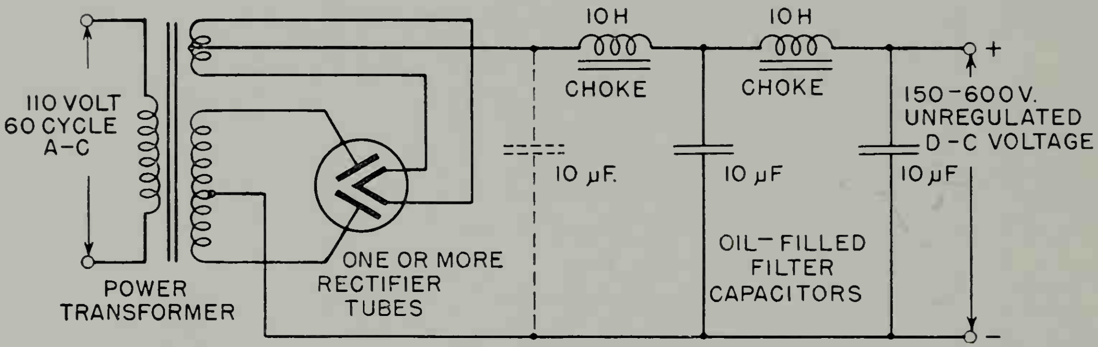

FIG. 7.22. Typical transformer-rectifier unit for a high-voltage power supply. The use of a capacitor input filter (dash lines) permits a higher output voltage and better filtering but increases the internal impedance of the unit. Generally speaking, vacuum-type rectifier tubes will give less trouble than gas-type rectifiers.

Each high-voltage power supply will, in general, consist of a *transformer-rectifier unit* and a *regulator unit.* A typical *transformer-rectifier unit* is shown in Fig. 7.22. The design of such units for given output voltages and currents is adequately covered in many textbooks and handbooks on electronics.[1]

The purpose of the *regulator unit* associated with practically every computer high-voltage supply is twofold. Specifically,

1. The regulator unit must keep the supply voltage in question constant within specified limits in order to minimize d-c amplifier drift due to supply voltage changes.

2. The regulator unit must keep the power-supply output incremental impedance (see Sec. 5.3) below a specified minimum value, not only

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[1] Reich, H. J., *Principles of Electron Tubes*, Chap. 11, McGraw-Hill, New York, 1941; Terman, F. E., *Radio Engineers' Handbook*, McGraw-Hill, New York, 1943.

throughout the range of expected computer signal frequencies, but also throughout a range of higher frequencies. This is necessary in order to avoid undesirable coupling between different amplifier stages and computing elements in the machine. Feedback effects due to such coupling might otherwise lead to errors in the computation and even to uncontrolled oscillations.

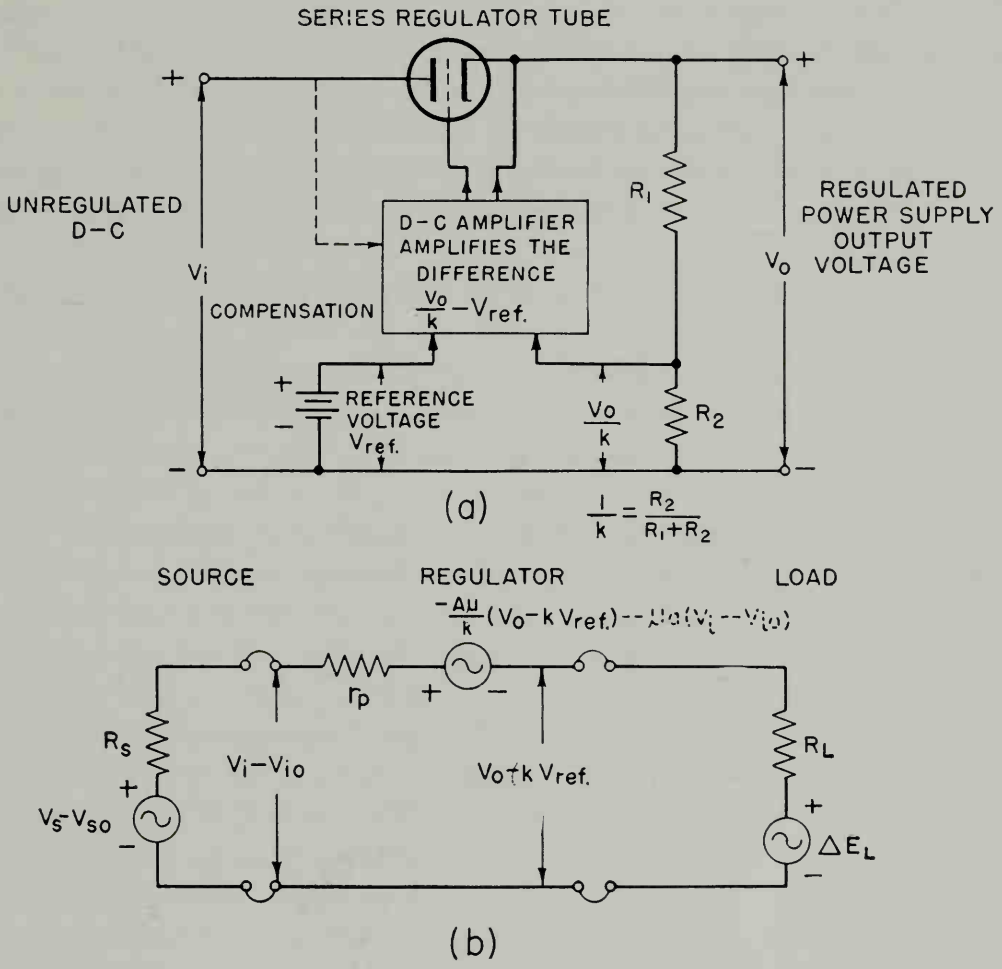
SERIES REGULATOR TUBE
+
UNREGULATED D-C
V_i
COMPENSATION
D-C AMPLIFIER AMPLIFIES THE DIFFERENCE $\frac{V_o}{k} - V_{ref.}$
R_1
REGULATED POWER SUPPLY OUTPUT VOLTAGE
V_o
+
REFERENCE VOLTAGE $V_{ref.}$
−
$\frac{V_o}{k}$
R_2
−
(a)
$\frac{1}{k} = \frac{R_2}{R_1+R_2}$
SOURCE
REGULATOR
$-\frac{A\mu}{k}(V_o - kV_{ref.}) -- \mu a(V_i -- V_{io})$
LOAD
r_p
+
−
R_S
+
$V_s - V_{so}$
−
$V_i - V_{io}$
$V_o - kV_{ref.}$
R_L
+
ΔE_L
−
(b)

FIG. 7.23. Block diagram and linear equivalent circuit of a degenerative electronic regulator unit. The grid voltage of the regulator tube is $\frac{A}{k}(V_o - kV_{ref})$ if no compensation is used. With compensation (dash lines) the grid voltage is $\frac{A}{k}(V_o - kV_{ref}) + a(V_i - V_{io})$.

The second function of the regulator unit is at least as important as the first one and is the principal reason for the almost exclusive use of *degenerative electronic regulators* in d-c analog computers. The solid lines in Fig. 7.23*a* show the block diagram of a degenerative electronic regulator. Such a unit is essentially a d-c power amplifier whose output plate current is taken as the supply current. Voltage feedback, through the resistors R_1

and R_2, tends to keep the power-supply output voltage V_o equal to a constant multiple kV_{ref} of the voltage V_{ref} where

$$\frac{1}{k} = \frac{R_2}{R_1 + R_2}$$

The voltage V_o/k is continuously compared with the reference voltage V_{ref}. The regulator amplifier tends to minimize the absolute value of the difference $(V_o/k) - V_{\text{ref}}$ by changing the effective resistance of the regulator tube, which is in series with the voltage source. The voltage feedback, will, moreover, tend to reduce the power-supply output incremental impedance.

Analysis of a Degenerative Regulator. The performance of a degenerative regulator with respect to voltage changes in the voltage source and current changes in the load can be analyzed conveniently by means of a *linear equivalent circuit* of the type discussed in Sec. 5.2. The solid lines in Fig. 7.23*b* show such a linear equivalent circuit for d-c conditions. The unregulated d-c source is indicated at the left; it has a resistive source impedance R_s. The source voltage V_s is capable of undesirable voltage changes $(V_s - V_{s0})$ with respect to its reference level V_{s0}. As demonstrated in Sec. 5.2, the regulator tube may be considered to act like a voltage source with the voltage ($-\mu$ times the grid voltage applied to the vacuum tube) and the internal impedance r_p, where μ and r_p are the amplification factor and the plate resistance of the regulator tube, respectively. The grid voltage of the regulator tube is $(A/k)(V_o - kV_{\text{ref}})$ as indicated in the block diagram of Fig. 7.23*a*; here $A < 0$ is the voltage gain of the d-c amplifier preceding the regulator tube proper. The load is characterized by its resistance R_L, which will be taken to include the effect of the resistors R_1 and R_2. The effects of current changes in the load are conveniently described by including in the equivalent circuit for the load a voltage source of voltage ΔE_L.

From the nodal equations of the equivalent circuit shown in Fig. 7.23*b* one finds, for d-c conditions,

$$V_o - kV_{\text{ref}} = \frac{(V_s - V_{s0})R_L + \Delta E_L(R_s + r_p)}{R_s + r_p + R_L\left(1 - \dfrac{A\mu}{k}\right)} \tag{7.5}$$

The internal impedance Z_o of the regulated power supply, with respect to voltage changes ΔE_L in the load, is, also for d-c conditions,

$$Z_o = -\frac{\partial \Delta E_L}{\partial(i - i_0)} - R_L = \frac{R_s + r_p}{1 - \dfrac{A\mu}{k}} \tag{7.6}$$

where $(i - i_0)$ is the deviation of the load current from its reference value i_0. Equations (7.5) and (7.6) show *that the greater the value of* $|\mu A/k|$ *the more the regulator will reduce the effects of source voltage changes as well as the internal impedance of the power supply.*

Compensation. The performance especially of low-gain regulators may be improved by adding a voltage proportional to the unregulated regulator input voltage V_i (see Fig. 7.23) to the regulator amplifier signal. Such a voltage is best introduced in the output stage of the regulator amplifier, as indicated by the dotted lines in Fig. 7.23*a*. The effective grid voltage change of the regulator tube is now

$$\frac{A}{k}(V_o - kV_{\text{ref}}) + a(V_i - V_{i0})$$

where V_{i0} is the reference level of the regulator input voltage V_i. The quantity $a < 0$ is the voltage gain or transfer function of the compensation circuit from the regulator input terminals to the grid of the regulator tube.

Through the use of the linear equivalent circuit, it is possible to derive expressions for the power-supply output voltage changes for d-c conditions

$$V_o - kV_{\text{ref}} = \frac{(V_s - V_{s0})(\mu a + 1)R_L + \Delta E_L[(\mu a + 1)R_s + r_p]}{(\mu a + 1)R_s + r_p + \left(1 - \frac{\mu A}{k}\right)R_L} \tag{7.7}$$

and for the power-supply output impedance (at direct current)

$$Z_o = -\frac{\partial \Delta E_L}{\partial(i - i_0)} - R_L = \frac{R_s(\mu a + 1) + r_p}{1 - \frac{\mu A}{k}} \tag{7.8}$$

for a regulator with compensation. It is seen that considerable improvements in regulator performance may be obtained by choosing values of the compensation gain a such that, approximately,

$$a = -\frac{1}{\mu} \tag{7.9}$$

The compensation gain a can be adjusted experimentally by observing the residual ripple on an oscilloscope connected to the regulator output terminals; the value of a is adjusted so as to minimize the ripple voltage. All regulators should be adjusted with the average rated load current flowing, so that all the tubes will have the correct operating voltages. The adjustment will have to be repeated whenever a tube in the regulator is replaced.

Practical Regulator Design. The application of these principles to practical circuits will be shown further below. *Much of the information given in Chap. 5 on the design, construction, and operation of computer d-c amplifiers will apply to the design, construction, and operation of the d-c amplifiers used in electronic regulators.* The input stages of high-quality regulator amplifiers should obtain their filament current from a regulated supply. The entire regulator constitutes a degenerative feedback loop, and its stability must be ensured by a judicious application of the principles presented in Sec. 5.4. Frequently, the gain of a regulator amplifier may be increased through the application of regenerative feedback in the manner discussed in Sec. 4.7. In applications where reduction of the long-time drift of the power-supply output voltage is of paramount importance, the use of self-balancing d-c amplifiers (see Sec. 5.6) will permit outstanding regulator performance. If all computer amplifiers are balanced frequently, however, or if self-balancing d-c amplifiers are used as computer amplifiers, long-time stability is not so important in a computer power supply as low output impedance and freedom from drift during the relatively short time of each computer run.

Reference Elements and Comparison Circuits. Gas discharge tubes such as neon bulbs and voltage regulator tubes are frequently used as reference elements, since they tend to maintain a fairly constant voltage across their terminals. This voltage does, however, tend to vary slowly as the tube ages and is also somewhat dependent on the ambient temperature and on the current through the tube. The effects of temperature changes can be minimized by operating each voltage regulator tube at a preferred current (about 12 milliamperes for a VR105[1]).

Batteries operated under conditions of low current drain are preferable to gas discharge tubes as accurate reference elements. It is not necessary or desirable to use standard cells as reference elements for computer power supplies; small batteries of the type used as B batteries in portable radios are capable of giving a voltage reference accurate to 0.05 per cent for a period of several months if the current drain is low (the grid of a vacuum tube is usually the only load) and the ambient temperature is kept reasonably constant.

Figure 7.24 shows two comparison circuits used in voltage regulators to obtain an amplified output voltage proportional to the difference between a fraction V_o/k of the regulator output voltage V_o and a given reference voltage. In the circuit of Fig. 7.24*a*, the reference voltage is applied to the cathode of a vacuum tube whereas the feedback voltage V_o/k is applied to the control grid. The differential amplifier circuit shown in

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[1] Greenwood, I. A., J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948.

Fig. 7.24*b* is preferable to the circuit of Fig. 7.24*a* because of its inherent compensation for the effects of variations in cathode emission and because the vacuum-tube plate current does not flow through the reference element.

Examples of Regulator Design for Computer Applications. Figure 7.25 shows a simple degenerative regulator suitable for use in both the positive and negative power supplies of a low-cost d-c analog computer. This regulator was widely used at the MIT Radiation Laboratory[1] and employs compensation to obtain an output impedance below 30 ohms at all frequencies from 0 to 100,000 cps. The output impedance for direct current is about 20 ohms with all reasonable transformer-rectifier units, and the regulation corresponds to a change in output voltage of less than 0.5 per cent for a 10 per cent change in line voltage.

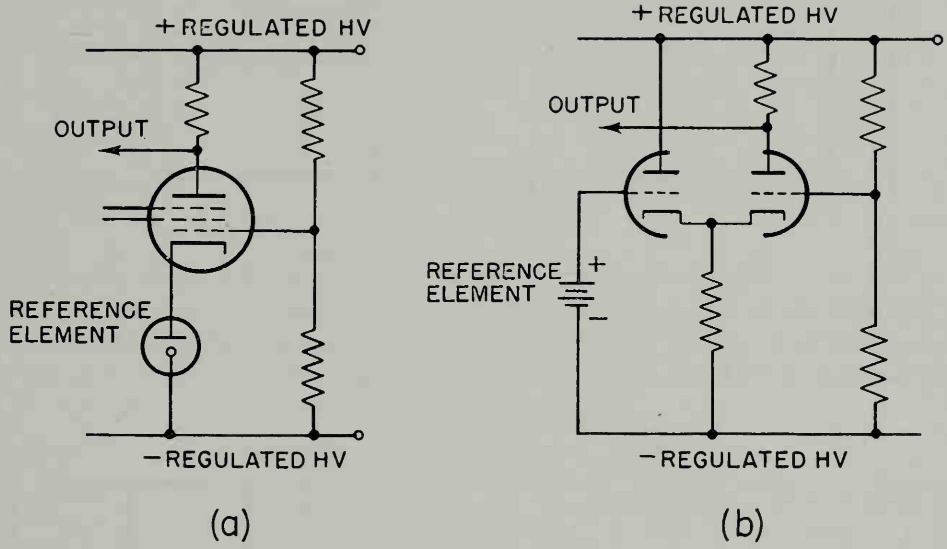

FIG. 7.24. Comparison circuits.

Figure 7.26 shows a high-quality regulator designed by Miller.[2] The difference between a fraction $(1/k)V_o$ of the output voltage V_o and the reference voltage obtained from a voltage regulator tube is amplified by two differential amplifier stages; the voltage regulator tube circuit may be replaced by a dry battery if desired. Compensation is introduced at one of the grids of the second stage. For best results, regulated direct current should be used to operate the filaments of the differential amplifier tubes. The output voltage of this regulator will, then, stay within 0.0002 per cent of the desired output voltage for line voltage changes of 12 per cent.[2] The d-c output impedance will be less than 1 ohm at direct

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[1] *Ibid.*

[2] Miller, S. E., Sensitive D-c Amplifier with A-c Operation, *Electronics*, November, 1941.

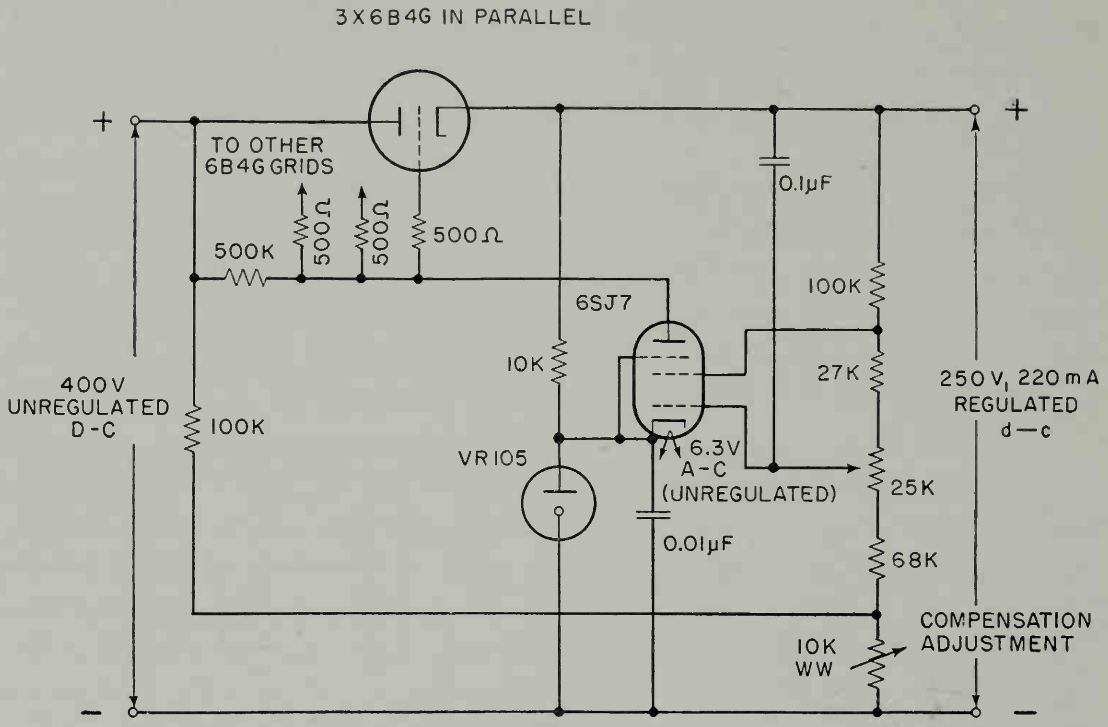

Fig. 7.25. Simple degenerative electronic regulator with compensation (I. A. Greenwood, J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948).

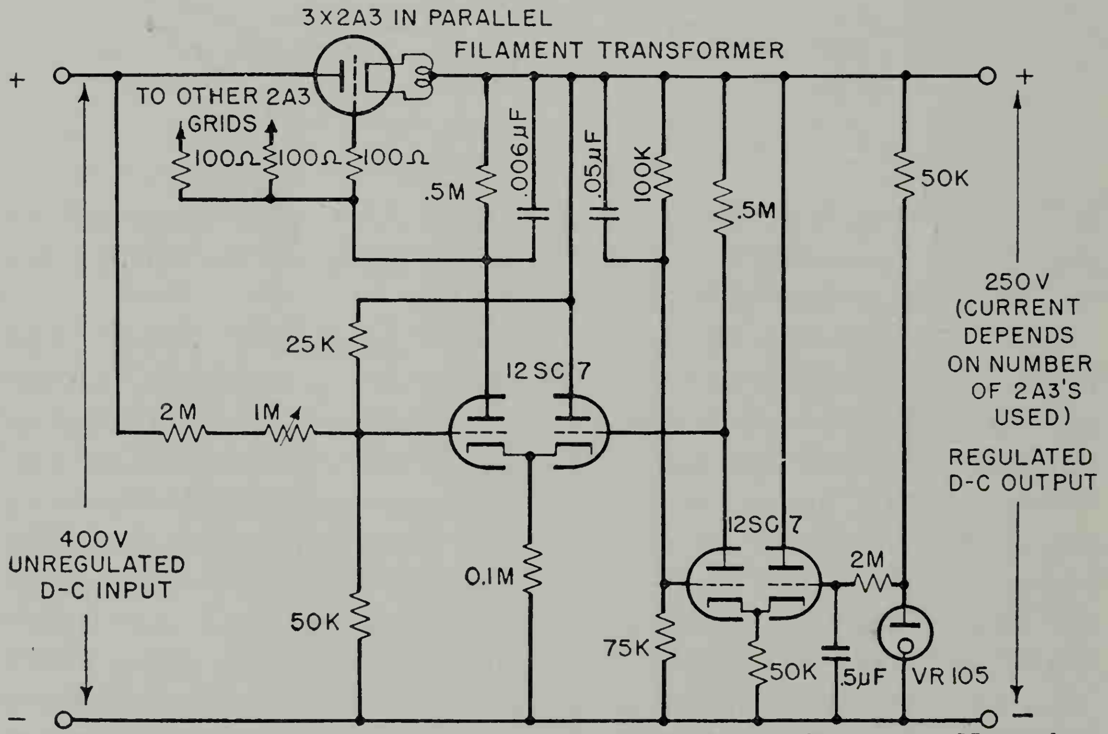

Fig. 7.26. Degenerative electronic regulator (S. E. Miller, *Electronics*, November, 1941). Regulated direct current should be used to operate the 12SC7 filaments. The 2A3 regulator tubes can be replaced by the more modern type 6AS7 if the correct bias level is arranged.

current with practical transformer-rectifier units. In order to keep the output impedance of such high-quality regulators low even at frequencies at which the regulator loop gain becomes low, some workers connect a capacitor of 4 to 20 μf across the regulator output terminals.

Computer Supplies with Common Reference Voltage. In d-c analog computers using several (positive and negative) high-voltage power supplies, it is advantageous to use a common reference element for all power supplies. The effects of changes in the reference voltage will then be at least partially canceled by the corresponding mutually opposing changes in the positive and negative supply voltages. Figure 7.27 illustrates a possible circuit for the use of a common reference element for a positive and a negative regulated power supply.

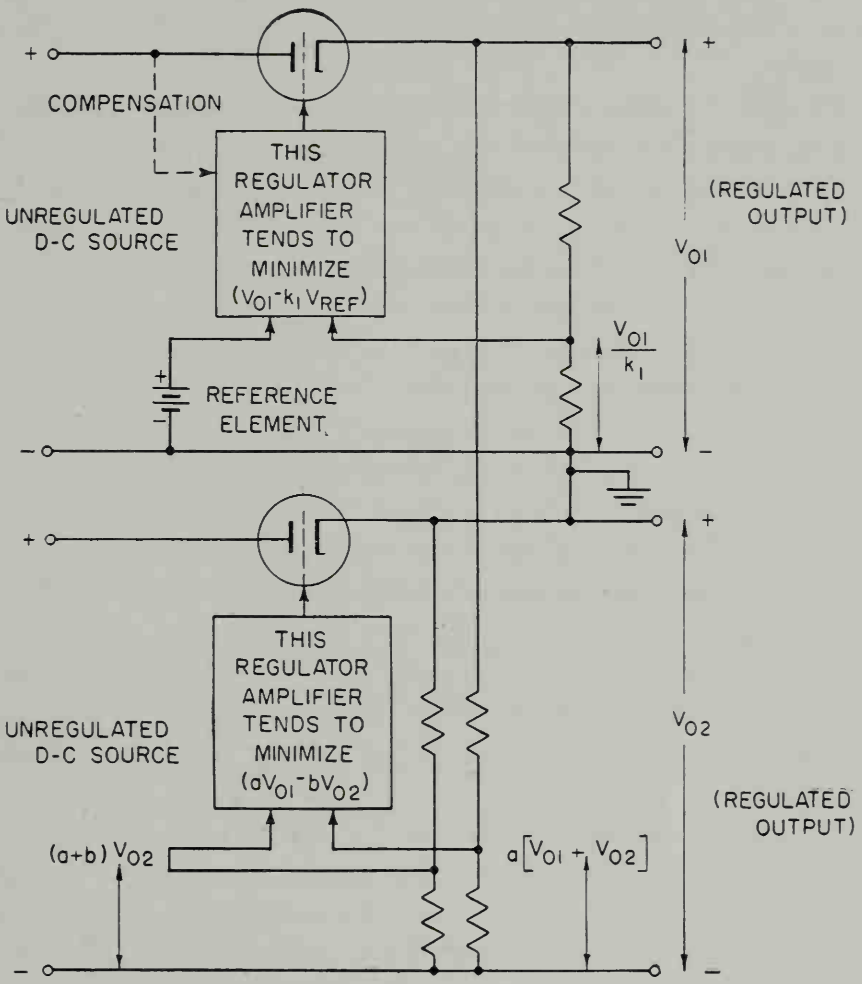

FIG. 7.27. Positive and negative computer power supplies regulated by reference to a single reference element.

The Construction of Electronic Regulators. As mentioned above, degenerative electronic regulators are essentially d-c amplifiers, and all the information given in Sec. 5.5 regarding the physical construction of such units and the choice of components will therefore apply here. Since regulator amplifiers may have high gains, care should be taken to avoid microphonics by shock-mounting at least the input tubes. If at all possible, the regulators should be mounted on a chassis separate from the transformer-rectifier units in order to minimize the danger of introducing 60-cycle hum into the regulators. Again, hum pickup as well as spurious feedback will be minimized by shielding the grid leads in the regulator input stages.

Positive and Negative Power Supplies for Computing Voltages. Every d-c analog computer must have provisions for supplying the original *input voltages* used in each computer setup. These voltages include the input voltages for function generators (some of these functions may be simply constant terms) and the initial-condition voltages applied to each integrator. It is customary to provide each d-c analog computer with terminals carrying voltages of plus and minus 1 machine unit (plus and minus 40 to 100 volts, as the case may be) for this purpose. These voltages may be obtained from other positive and negative high-voltage supplies having a common reference element only if the correct regulated voltages are available, since no voltage dividers may be used in this connection. For this reason, most multipurpose computers have special regulated power supplies or even battery sources for supplying these computing voltages. The power supplies in question may comprise regulator circuits essentially identical with those used in other high-voltage power supplies and designed with emphasis on good regulation and low output impedance; the load on each such power supply will be different for each computer setup.

The output voltage of one such power supply will serve as a *reference voltage* for the computer. The value of every variable voltage in the entire computer will depend on the reference voltage; but *the exact value of the reference voltage will, in general, not be too important,*[1] *since the values of all machine variables can be measured in terms of the value of the reference voltage.* All calibrations of recorder and meter scales as well as limiters and other function generators will, therefore, always be done in terms of fractions of the reference voltage.

Power Supplies for Operating Control Relays. Control relays (see Sec. 7.1), as well as solenoids used to brake motors or to operate clutches, are often operated on 24 to 28 volts direct current. Figure 7.28 shows two simple power supplies suitable for supplying direct current for control relays. Figure 7.28*b* shows a transformer power supply with bridge-connected selenium rectifiers and a ripple filter. Figure 7.28*a* shows a method of obtaining direct current directly from the 110-volt, 60-cycle a-c line by means of a high-voltage selenium rectifier; such rectifiers are now generally available commercially.

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[1] This is especially true in machines used only for the solution of sets of differential equations with constant coefficients.

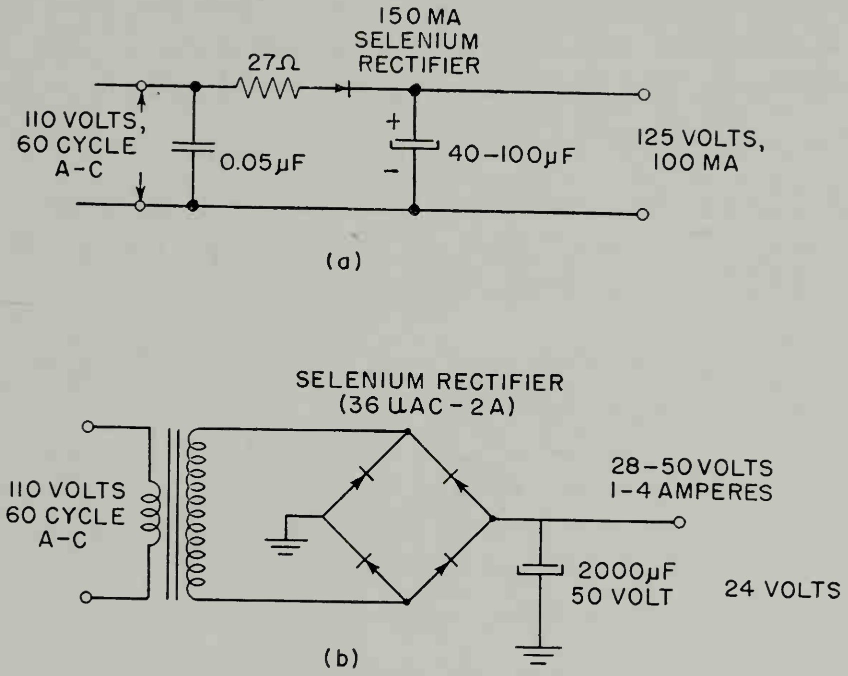

Fig. 7.28. Power supplies for d-c control relays.

If particularly fast relay operation is desirable, a relay may be fed an initial voltage surge of three to ten times the rated voltage by means of the resistor-capacitor circuit shown in Fig. 7.29.

7.5. Summary of D-c Analog-computer Operating Procedure. *Introduction.* The operating procedure for different d-c analog computers will

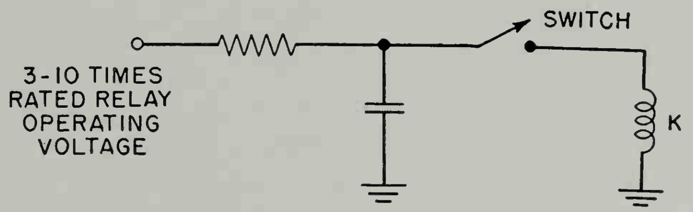

Fig. 7.29. Circuit for fast relay operation.

tend to vary depending on the design and function of the machine and especially on the design of the control circuits. Thus, in modern high-quality computers, many necessary control functions may be performed automatically and may not require the operator's attention at all (see Sec. 7.2).

In this section, a number of essential points common to the operating procedure of many d-c analog computers are summarized. This material refers mainly to flexible, multipurpose differential equation solvers of the "slow" d-c analog-computer type.

Placing a Computer in Operation. Before attempting to connect the computer to the power line by throwing the main power switch, computers employing simple switch-type control systems (see Sec. 7.2) require a check of the switch positions in both the power-supply and operating control systems. The operating manual should contain a *check list,* such as

> "Before placing the computer in operation . . . check the following switch positions: Filament Switch OFF; High Voltage Switch OFF; Reset-Compute-Hold Switch on RESET; Recorder Switch OFF . . . ."

In the case of intelligently designed push-button control systems, no such check will, in general, be necessary.

The computer is connected to the power line by throwing the main power switch (usually a wall switch) to the ON position. The filament voltage is then applied directly or by subsequently throwing an additional filament switch. After a proper time delay (which may or may not be enforced by a time-delay relay), plate voltage may be applied to the computer.

Permit the computer to warm up for at least 30 *min with filament and plate voltage* ON *before beginning computation.* This time may be used to set the computer up for new problems, if desired. In the following, it will be assumed that all scale factors have been tentatively chosen, that a tentative computer setup has been made in accordance with a block diagram as outlined in Chap. 2, and that all potentiometers, initial conditions, and limiting levels have been set up according to the block diagram.

Preparation for a Computer Run. After the high-voltage power supplies have been turned on, the power-supply voltmeters may be checked for correct indications. If the computer has d-c amplifiers which require manual balancing, each d-c amplifier must be balanced in turn by switching it into the BALANCE condition and by adjusting the balance controls until zero output voltage is read for zero input voltage. This balancing process should be repeated every half hour or, preferably, after every two or three computer runs. In the case of d-c amplifiers employing automatic balancing circuits (see Sec. 5.6), the auxiliary balancing controls may be touched up once a day.

After a final check of the connections (see Sec. 2.9), the computer is ready for the first test run. The computer is placed in the COMPUTE condition by operating the *compute* switch or push button. If the flashing of one or more of the overload or error-indicator lights indicates that the corresponding amplifiers are overloaded or not operating properly, changes in the computer setup or scale factors must be made to correct this situation.

When changing scale factors, one must be sure to record this fact on the block diagram. Do not forget to readjust all initial-condition potentiometers as well as all limiting levels and function generator settings corresponding to the new scale factors.

Recorder Adjustment. Depending on the type of recorder used (see Sec. 7.3), the recorder scale factor may be set initially, or the recorder may require calibration in the RESET condition. The recorder zero levels also require checking.

Computation. In general, each computer run is started by operating the *compute* switch or push button. The computer run may be terminated by placing the control switch in the RESET position or by actuating the *reset* or *stop* push button; or, again, the machine may return to the RESET condition through the action of an automatic timer. This procedure is repeated for each run. It may or may not be necessary to turn the recorders on and off separately for each run (see Sec. 7.2). In some computer laboratories, each run is duplicated in order to check precision and as an easy means for obtaining duplicate records.

7.6. Checking of Solutions. *Introduction.* Accuracy checks and estimates of the errors involved in analog-computer solutions are essential if the results obtained by the computing machine are to be relied on. The accuracy of the computed solution will depend intimately on the problem in question.[1]

It is frequently possible to compare a number of special cases of each set of analog-computer solutions with the corresponding correct solutions obtained numerically or analytically. Agreement of the analog-computer solutions for the special cases with the correct solutions will generally indicate a comparable accuracy in the case of the remaining analog-computer solutions unless the latter should be more sensitive to the effects of imperfections in the computing elements than the test solutions. This last question must be considered in the light of the nature of the solutions, and the special cases must be chosen judiciously.

Assuming, as will be most frequently the case, that the d-c analog computer is intrinsically capable of solving a given problem with the required accuracy and that the scale factors and the computer setup have been chosen so as to make the most of the capabilities of the computer, the validity of the solution will depend only on the proper functioning of the machine. It will then be useful to have methods for checking the computing elements for proper individual and combined operation.

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[1] Korn, G. A., The General Difference Analyzer, Mathematical Tables and Aids To Computation, 1951 (in preparation).

Some Checking Methods. Gunning and Mengel[1] suggest the following simple checks:

1. A simple minimum check consists in introducing suitable test voltages as initial conditions in all integrators of the completed computer setup and checking the output voltages of all amplifiers and function generators against computed values, with the machine in the RESET condition.
2. It is relatively easy to repeat a number of computer runs using a set of *new machine variables* (voltages) equal in value to those used in the original computation but having *opposite signs.* A comparison of the results of such check runs with the original results frequently tends to expose errors, particularly errors due to drift or saturation of d-c amplifiers in the computer.
3. Doubling the computing time by doubling the values of all integrating capacitances will tend to show up errors due to bad high-frequency response of one or more computing elements.

Checking by Resubstitution. Satisfactory results from the aforementioned simple checks will constitute necessary but not sufficient conditions for correct solutions. For a rigorous accuracy check, it will be best to utilize numerical methods which involve a resubstitution of the analog-computer solutions into the given mathematical relations. Such a method is particularly useful for checking solutions of sets of linear simultaneous equations (see Sec. 3.9).

In the important case of problems involving the solution of sets of ordinary differential equations, Gunning and Mengel advise *numerical checks or improvements of the machine solutions based on Picard's process of successive approximations* whenever this is possible. This process involves reducing the given differential equations to the form

$$px_i = f_i(x_1,x_2, \ldots ,x_nt) \qquad i = 1, 2, \ldots , n \tag{7.10}$$

Equations (7.10) are used to compute the functions px_i of the independent variable t from the solutions $x_1, x_2, \ldots, x_n$ obtained from the analog computer. Numerical integration[2] may then be used to compute each x_i from the corresponding function px_i; the new solutions may be used to *check* or *improve* the machine solutions. This method is useful also for problems so complicated that only an approximate d-c analog-computer solution is possible.

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[1] *Op. cit.*

[2] See H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry,* Van Nostrand, New York, 1946, for various practical methods of numerical integration.

CHAPTER 8

THE DESIGN OF COMPLETE D-C ANALOG-COMPUTER INSTALLATIONS

8.1. Complete D-c Analog-computer Installations: Composite Design Considerations. *Introduction: General Design Procedure.* Before the design of a computing machine is attempted, it will be necessary to note, preferably in writing, the purposes for which the proposed computer installation is to be utilized. This will help to determine

1. The time scale or scales to be used
2. The accuracy and dynamic range needed for each mathematical quantity
3. The type and number of mathematical operations to be performed
4. The types of associated equipment which might be used in connection with the computer; also special requirements as to power supply, size, weight, etc.
5. The amount of money, time, and effort which can be spent economically on the computer installation considering the type and number of the computations to be performed

These considerations may or may not point to the use of a d-c analog computer. In the first case, they will form the basis of a rough choice of the d-c analog-computing elements to be built; a knowledge of possible computing elements and their properties is, of course, essential to the designer.

As the design takes shape, various combinations of computing elements will be considered from the point of view of the stated requirements and reconsidered with respect to the possible choices of power supplies and other associated equipment. After a number of such "go-arounds," the composite design will have progressed to a stage where the design of individual components has to be considered in more detail.

Each finished design will have to be checked and will then have to undergo a final cost estimate; this procedure may have to be repeated until a desirable compromise is achieved.

Composite Design Considerations. Suppose that it has been decided that a d-c analog computer of the nonrepetitive type, say, is to be designed for a given set of applications.

The range of working frequencies, as determined by the types of expected solutions and the time scales used, together with considerations of cost and accuracy, will determine the choice of *multipliers* and *function generators* to be used.

The design of the *operational amplifiers* will constitute a significant portion of the over-all computer design. Decisions on the following questions, among others, will be required before the detailed design is considered:

A. D-c amplifiers

1. *Push-pull or single-ended?* Push-pull amplifiers will use more parts and consume more power than single-ended amplifiers but may save phase inverters and power supplies (see Secs. 5.5 and 8.7).
2. *How many stages? Use of regeneration?*
3. *Type of output stage?* This determines the amplifier output impedance and bears on the use of coefficient setting potentiometers as well as on the size and number of *power supplies* needed (see Sec. 5.3).
4. *Will automatic balancing circuits be used?*
5. *What quality of components is required?*

B. Computing networks

1. *What types of networks are needed? Will they be permanently installed or will they be interchangeable?* If permanently installed feedback networks are to be used in a multipurpose computer, the most useful combinations of summing and feedback resistors, integrating capacitors, etc., must be determined.
2. *Will coefficient setting potentiometers be used?* Coefficient setting potentiometers are convenient but may have to be driven by power amplifiers. Potentiometer loading may present a problem (see Sec. 4.1).
3. *Will precalibrated precision network components be used, or will the computer comprise a calibrating device?* The latter arrangement combines flexibility and low cost with a slight reduction in operating convenience.
4. *How will the computing networks be arranged physically with respect to the d-c amplifiers and the computer front panel* (see Sec. 8.3)?

Consideration of these questions will usually result in compromises between cost and accuracy. The designer is now in a position to choose

1. *Computer power supplies* (see Sec. 7.4)
2. *Control circuits* (see Sec. 7.2)
3. *Recording equipment* (see Sec. 7.3)

Finally, it will be necessary to specify the *physical construction* of the computer (see Sec. 8.2) and the *front panel layout* (see Sec. 8.3) with a view to convenience of operation and maintenance.

Discussion of Accuracy Requirements. Since it is not a simple matter to predict the errors due to the combined operation of several computing elements, the accuracy of a proposed or existing analog computer is usually specified in terms of the minimum "component accuracies" of its individual computing elements.

The accuracy of each d-c analog-computing element may be specified in volts, in machine units, or by percentage errors usually referred to the full-scale output voltage of the computing element in question. Thus, high-quality d-c analog computers may have component accuracies between 0.2 and 0.01 per cent of full scale.

The accuracy of the final results obtained with an analog computer having a given component accuracy depends on the problem in question and on the computer setup and scale factors used. In solutions of linear differential equations, for instance, better accuracies can usually be expected if the solutions are well-damped oscillations rather than unstable oscillations.

Speaking very generally, it may be said that solutions having an accuracy of 0.3 to 2 per cent are usually considered satisfactory with high-quality d-c analog computers handling nonlinear problems.

A method for obtaining *upper bounds* for the errors to be expected from d-c analog computers as well as a discussion of the relative importance of various error sources in d-c analog computers is available in the literature.[1]

The "linear" computing elements (networks and operational amplifiers) are the potentially most accurate computing elements of a d-c analog computer. When specifying the accuracy of these components, *one must remember that the over-all accuracy of the computer is usually already limited by the accuracy of the multipliers and function generators used.*

The accuracy required of a d-c analog computer will be determined by the type of application to which it is put. Some types of problems, such as ballistic trajectory computations (see Sec. 3.4), require relatively accurate numerical results. Other types of problems, such as certain stability problems in automatic control engineering, may require information only as to the *type* of solution.

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[1] Korn, G. A., The General Difference Analyzer, *Mathematical Tables and Aids to Computation*, 1952 (in preparation).

Various methods of *checking* computer solutions are discussed in Sec. 7.6.

Recommended Design Practice. Four important criteria for appraising the finished design of a d-c analog computer are:

1. *Accuracy and reliability.*
2. *Operating convenience.* As a rule, no highly technical training should be required for the operation of a d-c analog computer.
3. Initial and operating *costs.* Relatively low initial and operating costs are among the principal advantages of properly designed d-c analog computers.
4. *Easy maintenance.* This is an important factor in determining economical operation.

The various components making up the computing machine must be carefully chosen and modified so as to work together as part of an integrated design. Nevertheless, individual chassis units should be capable of operating independently of other components as far as possible. If this rule is observed, it will be easy to test and service individual components. It will also be relatively easy to make design changes at a later date; *e.g.*, it will be possible to replace a computer amplifier by a later, improved model without changing or adjusting other components.

Again, similar components or chassis units should be easily interchangeable so that unused components can be used as spares.

8.2. Construction, Wiring, and Installation. *Mechanical Construction: Chassis and Cabinets.* The emphasis in the mechanical construction of any computer should be placed upon *ruggedness* and *accessibility for easy maintenance.* In order to fulfill the latter requirement of "Design for Maintenance," the various computing elements should be built up on their metal chassis in such a manner that individual units can be pulled out separately. It is possible to build many special arrangements which permit units to be slid or rolled out on special rails or folded out on hinges so that their undersides can be easily inspected. But the designer should be careful not to go overboard in this respect. In the majority of cases, analog computers should be constructed, wherever possible, of commercially available components.

With these considerations in mind, one may say that standard enclosed relay rack construction will, in general, be a good choice at least for low- to medium-cost installations. In this mode of construction, the computing elements are built on metal chassis in sets of one or two. The chassis are in turn supported by standard relay panels which also carry the necessary controls and jacks. The panel assemblies are placed into standard relay racks. Individual units can be easily removed for inspection and maintenance. For easier removal and replacement of the individual units, it is helpful to modify the relay racks as shown in Fig. 8.1 by screwing simple angle irons to the sides of the relay rack cabinet to support the chassis, which can then be slid in and out of the cabinet. This procedure is further facilitated by placing convenient handles on the front of each rack panel.

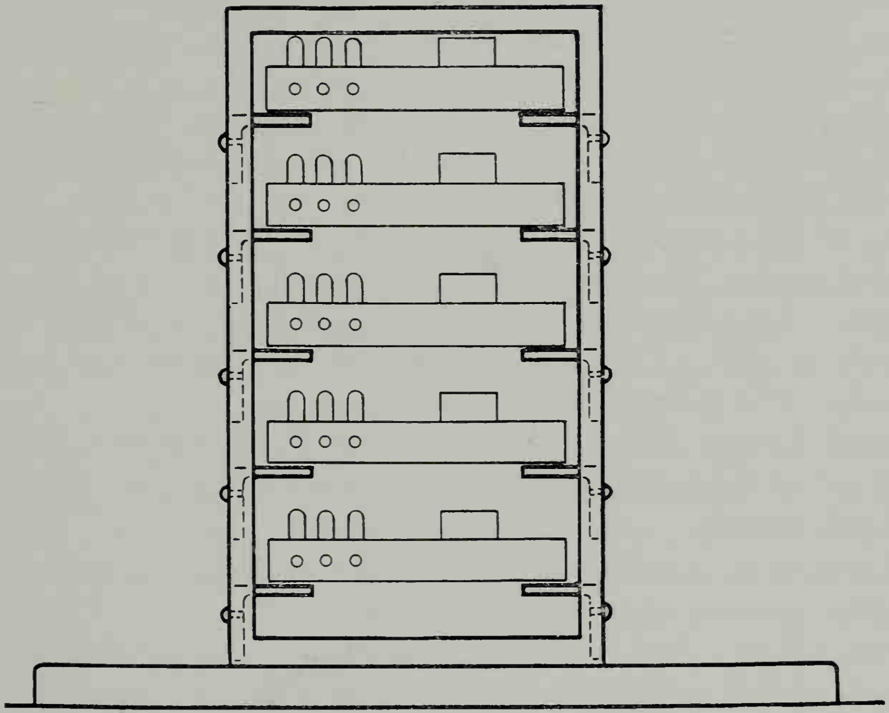

FIG. 8.1. Rear view of standard relay rack modified by the addition of angle irons which permit individual units to be slid in and out.

If a lightweight mobile installation is desired, aircraft-type racks and chassis may be used together with miniaturized electronic circuits. Such construction makes it easily possible to mount an entire analog-computer installation on a trailer or truck.

At least in larger computers, it is good practice to place the power supplies in a cabinet or rack separate from the computing elements proper in order to avoid hum due to the magnetic fields of the power transformers. Servomechanisms using a-c motors may also be housed in separate cabinets together with their associated power supplies.

The construction of *electromechanical assemblies* and of d-c amplifiers used in d-c analog computers has been treated in Secs. 6.4 and 5.5, respectively.

Wiring Practice. No specialized wiring techniques are required for the construction of d-c analog computers. Some emphasis should be placed on rugged wiring, so that slight shocks and vibrations cannot change circuit constants. As a matter of course, reasonable safety precautions should be observed, and all wiring and fusing should be done according to the Underwriter's Laboratories' specifications.

In order to save time and labor, it is well to give some thought to "production engineering" techniques when a number of identical amplifiers or other computing elements are to be wired. Such techniques include the use of subpanel terminal strips for mounting resistors. These and similar subassemblies can be planned and constructed before the complete chassis in question is assembled.

Choice of Components. As far as possible, only high-quality components should be used in the computer construction; the joint Army-Navy (JAN) specifications may be used as a general guide. Good discussions of the properties of various components are contained in the MIT *Components Handbook.*[1] The problem of selecting certain critical components for computing elements involving d-c amplifiers has been treated in Sec. 5.5. The choice of components for computer *power supplies* is discussed in Sec. 7.4.

Interunit wiring. All wiring interconnecting the various component units of a d-c analog computer usually falls into one of the following general classifications:

1. Common *ground straps* for all units. *It is not good practice to rely on chassis grounds in d-c analog computers.*
2. Filament and high-voltage *power-supply wiring* for electronic components. High-voltage power-supply wiring in common to more than one amplifier should preferably have an impedance of less than 1 or 2 ohms in order to minimize coupling between amplifiers.
3. *Control wiring* for making connections to the reset and hold relays on integrators and to other relays controlled by a common timer, etc., as well as wires leading to special overload indicators.
4. *Metering system wiring* may be necessary in computers possessing a central system for checking, balancing, and calibration.
5. *"Signal" wiring* carrying the computer voltages (machine variables) proper between computing elements. In flexible multipurpose computers, the input and output terminals of the computer are brought out to jacks on the computer front panel and patch cords are used for flexible interconnections. The use of shielded patch cords is recommended.

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[1] Blackburn, J. F., *Components Handbook,* MIT Radiation Laboratory Series, Vol. 17, McGraw-Hill, New York, 1949.

All connections between the chassis units and the interunit wiring should be made by means of good, plug-in connectors which must permit positive connections as well as easy removal of individual chassis units. The interunit wiring may be cabled up along the rear corners of the relay rack cabinets and connected to multicontact receptacles attached to the cabinet walls. Each individual removable chassis unit may then be connected to a receptacle by means of a short individual cable attached to the chassis and terminated in a *male* plug-in connector.

In the interest of simple design and easier maintenance, it is desirable to keep permanent wiring between individual chassis units to a minimum.

Installation. While the spirit of research may be dominant in a computer laboratory, it is sometimes wise to remember that a finished computing device is no longer a breadboard. It is absolutely necessary to observe all safety regulations specified by the Underwriter's Laboratories with respect to mechanical as well as electrical installation of the computing equipment. The floor beams of some office buildings may not be able to support the concentrated weight of relay racks containing heavy power supplies. In such cases it is best to place the computer cabinets on a low wooden platform which will distribute their weight. Interconnections between different computer cabinets can be very conveniently arranged underneath such a platform.

All power connections to the computer except possibly for service outlets should be through a main wall switch containing the necessary fuses and circuit breakers. Instead of relying on the high-voltage indicator lights for the protection of maintenance personnel, it may be useful to provide a green safety light which is on only when the main wall switch is *off*.

Sufficient space should be allowed in front of the relay rack units for unhampered operation and in back of these units for convenient maintenance. If at all possible, the room containing the computer should be devoted to the computer and associated equipment alone, so that the operating personnel will not be disturbed unnecessarily in their somewhat complex task. Sound insulation of the ceiling also helps in this respect. The computing room might also contain a desk, blackboard, drawing board, and a bulletin board. A large table on which to unroll and examine computer records is helpful. It may be desirable in some installations to place the recorder on a rubber-tired cart which can be rolled into position wherever the operator happens to be working. A remote-control station of the type described in Sec. 7.2 will permit the operator additional freedom by allowing him to control the computer from any point in the room.

A large computer installation will generate an appreciable amount of heat, so that some attention will have to be given to the air conditioning of the computer room. The accuracy of high-quality installations will be improved if the ambient temperature is kept reasonably constant (within about 10°C) by a thermostat controlling the air-conditioning system.

Aside from purely technical considerations, the careful planning of a pleasing and efficient electronic computer installation will be repaid in terms of more intangible assets. A computer installation may be made a show place and, as such, may prove to be of considerable sales value to the organization operating it.

8.3. Front Panel Layout. *Introduction.* Generally speaking, the front panel of a computing machine will bear the following types of components:

1. *Operating controls* or control and indicating devices enabling the operator to initiate, supervise, and terminate the operating functions of the computer.
2. *"Adjustment" controls* to balance d-c amplifiers and to adjust them and other computing elements for optimum operating conditions.
3. *Controls for adjusting variable parameters and functions* which are required for setting up the specific problem at hand.
4. Means for *interconnecting* the computing elements for each computer setup. Sets of telephone jacks for plugging in patch cords are usually used.

In other words, the components on the front panel serve three functions: computer operation, computer adjustment, and computer setup. An elegant and well-thought-out panel design which permits the operating personnel to accomplish these functions easily and conveniently is a very important consideration in the design of the complete computer. As a matter of fact, if the computer is properly designed, the front panel will be the only part of the computer with which the operating personnel will come in contact during the vast majority of the time the computer is being used. *For the convenience of the operating personnel, the controls performing the various functions outlined above should be separated physically* in such a manner as to preclude errors caused by mistaking one component for another.

Operating Controls. The computer operating controls should be as convenient as possible to use, and their functions should be clearly indicated for psychological as well as for technical reasons. It is usually a good plan to have all the control switches, push buttons, and indicator lights together on one separate panel where the operator can see them easily. An exception is often made in the case of overload lights, which may be mounted on the same panel as the corresponding d-c amplifiers.

Finally, it is highly advisable to bring the computer start, compute, stop, reset, and hold controls out to a small *remote-control box* by means of a sufficiently long cable plugged into the computer proper. Such a remote-control box, illustrated in Fig. 8.8, can be conveniently carried in one hand and will enable the operator to start and control the computer from any position while he is changing the setup, adjusting parameters, checking the recording equipment, or making calculations on the blackboard. The small remote-control box does not have to carry indicator lights, but the lights on the computer panels should be large enough to be readily seen from any position in the laboratory.

Adjustment Controls. Modern design practice is eliminating from the front panel all but those controls which are frequently used during the checking and operation of the computer. Potentiometers used to adjust power-supply voltages as well as amplifier feedback controls usually have to be adjusted only rarely, such as when tubes are changed, and therefore do not belong on the front panel. The same applies in many cases to the gain and damping controls of computer servomechanisms.

The *balancing controls* formerly associated with every d-c amplifier are also being eliminated from the front panel to some extent since the advent of modern automatic balancing devices (Sec. 5.6).

It is, then, possible to build amplifier panels *without any external controls whatsoever* and to introduce considerable simplifications in front panel design. If supplementary balancing controls do appear in connection with automatic balancing devices, they will require adjustment only rarely and therefore should be only screw-driver controls. In general, it is important that all balancing and similar adjustment controls either be provided with locking nuts or be recessed under the panel in such a manner that they cannot be accidentally or inadvertently disturbed.

Parameter Adjustment Controls. The most important adjustment controls used in setting up a computer for a given problem are the *constant coefficient setting potentiometers* discussed in Sec. 4.1. It is important that these controls be arranged within easy reach of the eyes and hands of the operator, that the dials be sufficiently large and clearly labeled to be set up easily, and that each dial have a clearly visible label associating it with its number in the block diagram. These provisions will greatly aid in reducing errors caused by confusing one dial with another, an all too common occurrence. The coefficient setting potentiometers and their associated dials may be arranged on separate relay panels in groups of five, ten, etc., as desired.

Another group of controls are those used to adjust functions to be obtained from various types of universal function generators (Secs. 5.6 and 6.5). Such controls will be placed on the front panel if they require frequent readjustment between computer runs. It is again important that all these controls be arranged so that they cannot be accidentally disturbed, *e.g.*, by passing patch cords over them.

Patch Bays. In flexible, multipurpose computers with which we are chiefly concerned here, the computing elements are connected in accordance with the setup by means of patch cords. These patch cords may be of the ordinary telephone variety, but it is desirable to have the shielded type in order to prevent cross talk between adjacent cords.

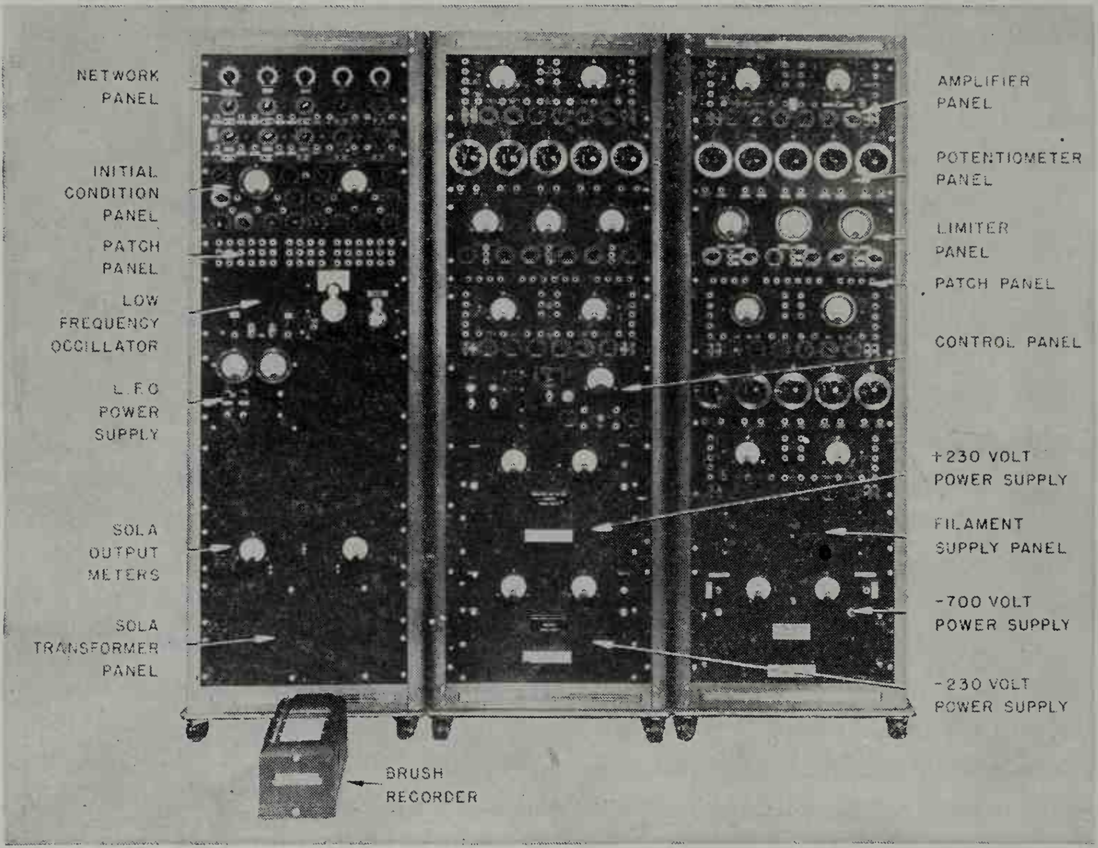

FIG. 8.2. Front panel view of Sperry Gyroscope electronic analog computer (Sperry Gyroscope Company, Great Neck, N.Y.).

There are two differing philosophies concerning the arrangement of the jacks used in connection with these patch cords. One of these philosophies holds that connections to each computing element should be made by means of jacks on the individual panels. The Sperry Gyroscope computer shown in Fig. 8.2 is an example of this type of construction.

The second design philosophy requires that the connections from the various computing elements be brought out to a large central patch bay resembling a telephone switchboard. This second arrangement makes the front panel look somewhat neater and keeps the patch cords from hanging over controls, dials, etc., associated with various computing elements. Such designs necessitate rather complex wiring between the patch bays and the computing elements. The price of such an arrangement is paid not only in terms of increased installation and maintenance costs but also in terms of cross talk between the wires. In the authors' opinion, this practice is frequently objectionable, especially in view of the fact that the modern types of d-c amplifiers do not require frequent adjustments. Instead, very satisfactory and pleasing arrangements involving jacks on the computing elements themselves can be obtained as shown further below.

Front Panel Designs. The CURTIAC computer shown in Fig. 8.10 is a good example of the separate patch-bay type of construction. Note the very neat appearance of the computer and the overload lights collected along the upper edge of the computer which are connected to each unit by interconnecting wiring. In addition to the overload lights, there is also a circuit causing a buzzer to sound whenever any one of the amplifiers overloads and, mercifully, a switch to cut the buzzer out of the circuit.

Whereas the Sperry computer (Fig. 8.2) does suffer from patch cords overhanging the controls, a new design shown in Fig. 8.3 combines the advantages of both the design philosophies outlined above. Since the d-c amplifiers do not need frequent adjustments, it has become possible to arrange the summing amplifier and integrator panels to form a patch bay in spite of the fact that all patching connections are made directly to the computing elements themselves. The potentiometer panels, which must be readily accessible during and between computations, are kept free from patch cords by being located at both sides of the d-c amplifier panels. Thus a convenient area for patching is formed in the center of the computer front panel. A control panel is shown in a convenient position at the *left*, and a remote-control station is indicated.

Figure 8.4 illustrates a modification of this arrangement. Here actual block-diagram symbols have been drawn on the front panels of some of the computing elements for even greater ease in checking computer setups. In both Figs. 8.3 and 8.4, the overload lights associated with each d-c amplifier are shown mounted directly on the respective amplifier panels.

A considerable portion of the total time spent in using any automatic computer is devoted not to actual computation but to the evaluation of solutions and to computer setups for new problems. In order to reduce the computer "dead time," it might be desirable to utilize the computing elements in two (or more) computer setups on a rotation schedule. A system of automatic interconnections in the manner of the MIT differ-

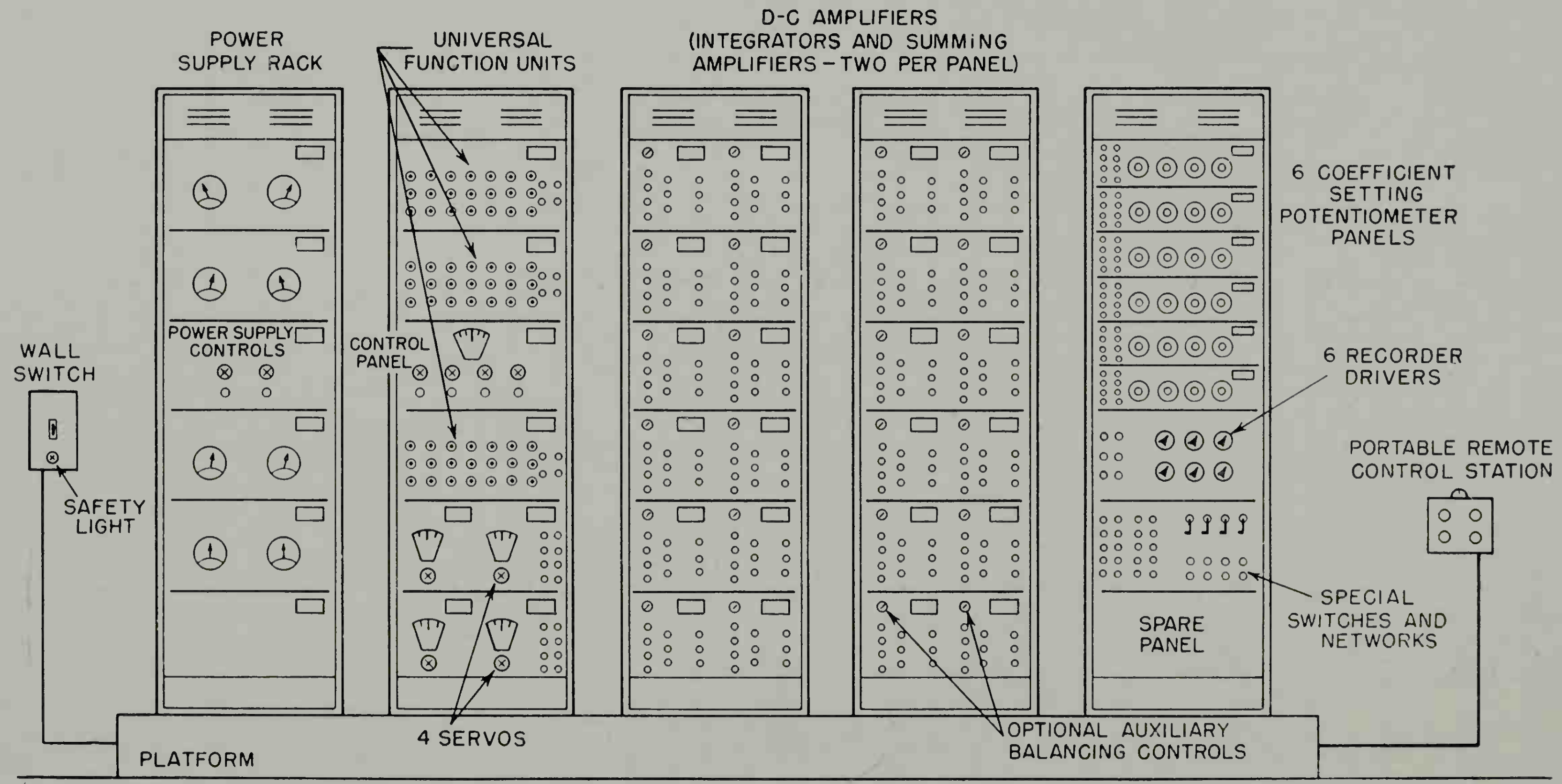
POWER SUPPLY RACK
UNIVERSAL FUNCTION UNITS
D-C AMPLIFIERS (INTEGRATORS AND SUMMiNG AMPLIFIERS - TWO PER PANEL)
6 COEFFICIENT SETTING POTENTIOMETER PANELS
6 RECORDER DRIVERS
PORTABLE REMOTE CONTROL STATION
WALL SWITCH
POWER SUPPLY CONTROLS
CONTROL PANEL
SAFETY LIGHT
SPECIAL SWITCHES AND NETWORKS
SPARE PANEL
4 SERVOS
PLATFORM
OPTIONAL AUXILIARY BALANCING CONTROLS

FIG. 8.3. New front panel design for a medium-size d-c analog computer. Note the labels on each panel unit. The jacks on the computing elements are arranged to form a patch bay in the center. Recorders are not shown.

ential analyzer (described by Caldwell and Bush)[1] is too complicated and expensive for most installations. It should, however, be possible to provide so-called *removable patch panels* of the type used in the IBM punched-card machines. This technique is best adaptable to computers having separate patch bays and may indeed of itself give rise to a certain amount of cross talk. Cross talk may become serious if frequencies in excess of 3 cps are involved in the computation unless careful attention is given to shielding.

Several duplicate "patch boards" may be plugged into the jacks of the main patch bay. The actual setups are made on the removable patch boards; this can be done conveniently on any table while the computer is being used for a different problem.

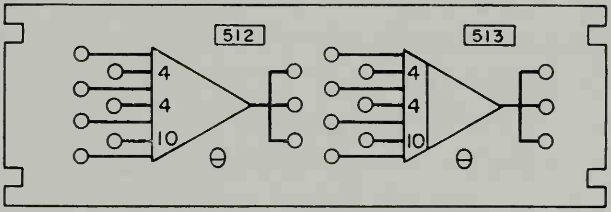

FIG. 8.4. Amplifier panel with engraved block-diagram symbols.

If the correct coefficient and function settings for each problem are made, several problems may be set up on different patch boards and run in rapid succession. This would permit a more economical utilization of the computer. This principle is generally applied to punched-card computers and has been applied to a d-c analog computer in the very elegant installation of the RAND Corporation[2] (see Sec. 8.7) as well as in some computers built by the Reeves Instrument Corporation[3] (see Fig. 8.9).

A particularly simple arrangement of this type results if the (relatively cheap) summing and feedback networks for each problem are set up on the removable patch boards. In this case, only the input and output terminals of d-c amplifiers, potentiometers, multipliers, and function generators appear on the main patch bay.

Finally, in order to allow for the use of special networks or circuit elements and for rapid circuit changes, it is very useful to have one or two special panels bearing rows of interconnected jacks and a number of double-pole–double-throw switches with their terminals brought out to other jacks.

8.4. Low-cost Computer Installations. *Introduction.* Low-cost d-c analog computers and, in particular, small differential equation solvers can find a vast number of uses in universities and small industrial laboratories for purposes of instruction and analysis. In larger computer laboratories, small computers will be found useful for making *exploratory computations* in order to permit more economical utilization of more accurate computing machines.

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[1] Caldwell, S. H., and V. Bush, A New Type of Differential Analyzer, *J. Franklin Inst.*, **240**: 255, 1945.

[2] Santa Monica, Calif.

[3] New York, N.Y.

Again, d-c analog computers of fairly low accuracy are often entirely sufficient for analyzing the stability of servomechanisms in cases where only roughly quantitative estimates of parameter values necessary for stable operation are required.

If a d-c analog computer is to be designed with a view to low cost, the following factors should be taken into consideration:

1. *The d-c amplifiers should use as few vacuum tubes and other components as possible.* Triode circuits are often simpler than pentode circuits. The use of regeneration may be helpful for obtaining the required gain with as few stages as possible.

2. *The d-c amplifiers should require only two high-voltage power supplies.* This consideration indicates the use of differential or push-pull amplifiers and/or cathode-follower output stages.

3. *The cost of summing and feedback resistors and capacitors must be kept down as far as the required accuracy will permit.* A plug-in arrangement for the summing and feedback resistors and capacitors will reduce the total number of such components needed. *Low-cost resistors, capacitors, and also potentiometers can be used for the computing networks if a simple comparison bridge is available for calibration purposes.*

4. *The power requirements of the d-c amplifiers should be as low as practicable.* The power output required of each d-c amplifier is reduced materially if constant coefficients are set up through a choice of fixed or variable feedback resistors, as shown in Fig. 4.4, rather than through the use of coefficient setting potentiometers which tend to load the d-c amplifiers. The calibrating bridge mentioned above could be used to set the coefficient values.

5. The control circuits, patching arrangements, etc., should, of course, be kept as simple as possible. The machine must, however, still be reliable and convenient to use; otherwise, it may just be shelved as a scientific curiosity.

Low-cost d-c analog computers may merely be sets of individual two- or three-tube operational amplifiers operated from a common power supply and interconnected by means of simple patch-cord systems.[1] The Aeronautical Research Center of the University of Michigan[2] operates several small-sized table-top d-c analog computers. One of them, described by McDonald,[1] comprises a number of small two-tube amplifiers using the circuit shown in Fig. 5.31 and neatly plugged into a large chassis. A front panel on this chassis serves as the patch bay. Various operational amplifier arrangements can be built up by means of plug-in resistors, and patch cords are used to interconnect different operational amplifiers into computer setups.

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[1] Ragazzini, J. R., R. H. Randall, and F. A. Russell, Analysis of Problems in Dynamics by Electric Circuits, *Proc. IRE*, May, 1947, pp. 444–452.

[2] Willow Run Airport, Ypsilanti, Mich.

The excellent report by Hagelbarger, Howe, and Howe[2] on the utilization of another table-top analog computer at the Aeronautical Research Center of the University of Michigan shows page after page of remarkably accurate results from dozens of elegant applications of this small machine (see also Chap. 3).

The Boeing Electronic Analog Computer (*BEAC*). An electronic analog computer of high utility and dependability but costing only about $7,000 was developed by the Physical Research Unit of the Boeing Airplane Company[3] in connection with an Air Force contract and proved to be very useful to various groups in the company.

A front panel view of the Boeing analog computer is shown in Fig. 8.5. A standard relay rack has been converted into a "skeleton rack" bearing the plus and minus 300-volt regulated power supplies, a central vertical patch panel, and a control panel. On the left and right of the central patch panel, there are 27 panel spaces which may be used to mount computing elements arranged on small subpanels and connected to the power-supply wiring by means of cables with plug-in connectors.

The standard complement of computing elements for the Boeing analog computer includes 12 regenerative d-c amplifiers, one isolated double-diode limiter of the type shown in Fig. 6.47*d* and described in Section 6.7, 24 coefficient setting potentiometers of the helical type, and a "volt box" containing a dry battery used as a power supply for computing voltages. The control panel carries the compute-reset switch as well as a balancing meter and a voltmeter for general use.

The d-c amplifiers of the Boeing analog computer use the elegant regenerative circuit shown in Fig. 5.32 and described in Sec. 5.5. Automatic balancing units for use with the Boeing d-c amplifiers are available as optional equipment.

The amplifier output is said to be sufficient to drive a Brush recorder pen motor *directly* with fair accuracy.

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[1] McDonald, D., Analog Computers for Servo Problems, *Rev. Sci. Instruments*, **21**: 154, 1950.

[2] Hagelbarger, D. W., C. E. Howe, and R. M. Howe, *Investigation of the Utility of an Electronic Analog Computer in Engineering Problems*, UMM-28, Apr. 1, 1949.

[3] Seattle, Wash.

On each amplifier subpanel, five 1-megohm summing resistors are permanently connected to each d-c amplifier input terminal. The amplifier input and output terminals are also brought out to jacks on the subpanel so as to permit one to make up various operational amplifiers by plugging combinations of resistors and capacitors into the jacks on the subpanel.

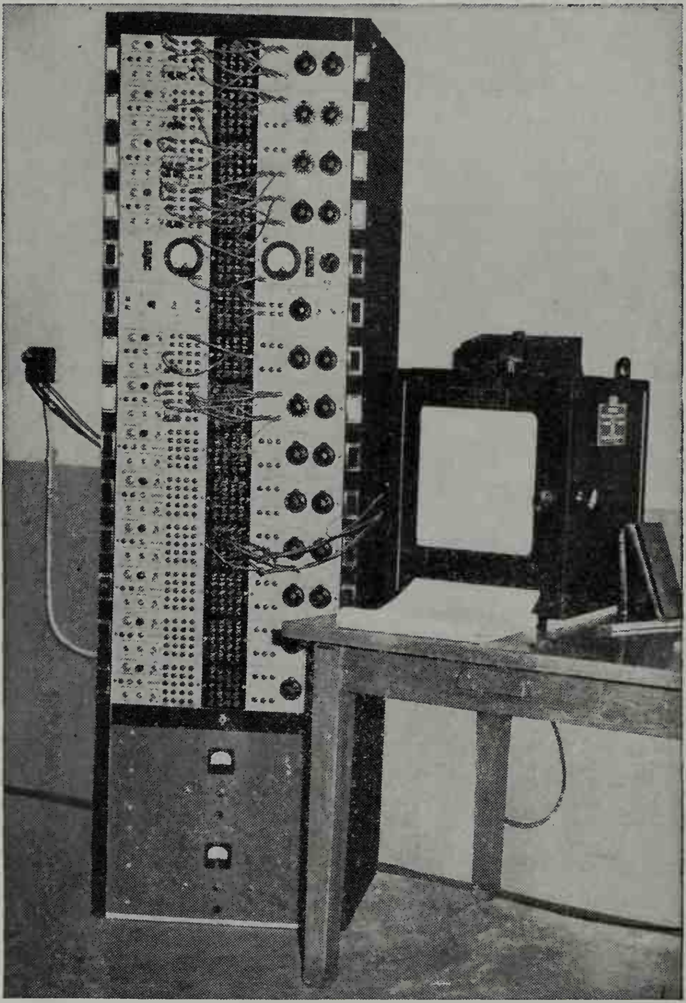

FIG. 8.5. Front view of the Boeing electronic analog computer.

Each amplifier subpanel also bears two balancing potentiometers, a balancing switch, and a neon overload light. The balance meter is situated on the control panel.

Each d-c amplifier is provided with a quick-acting reset relay of the quick-acting millisec type. Initial-condition voltages must be obtained from one or more volt boxes used as reference power supplies. The central patch panel is not permanently connected to any of the computing elements but merely contains 13 buses running from top to bottom of the cabinet. These buses may be utilized instead of long patch cords for interconnecting computing elements located at different levels on the front panel. The buses of several racks may be interconnected if the individual units are to be combined into a larger computer installation.

The good high-frequency response of the all-triode d-c amplifiers used in the Boeing computer permits the use of the machine as a *repetitive computer* at eye-persistence repetition rates as well as for operation at reduced time scales.

8.5. Repetitive Computers. *Introduction.* The principles of the repetitive d-c analog computer have been discussed already in Sec. 1.1. A repetitive d-c analog computer is an electronic differential analyzer capable of working on a time scale so fast that complete solutions are generated within a fraction of a second. Repetitive computers moreover contain provisions for resetting the machine variables to their given initial conditions periodically at repetition rates between 30 and 100 cps. In this manner, computer runs yielding solutions of the given sets of ordinary differential equations are repeated periodically; each solution can thus be displayed on the screen of a cathode-ray oscillograph whose linear sweep is synchronized with the initial-condition setting circuits of the computer. It is also possible to use a cathode-ray oscillograph for displaying *parametric plots* or graphs of one machine variable against another by connecting the horizontal and vertical input terminals of the oscillograph to the proper terminals in the analog computer. The display methods possible with repetitive computers open up striking possibilities for continuously observing the effects of changes in parameters and initial conditions on the solutions of sets of differential equations. Examples of the many useful applications of this technique are:

1. Rapid approximations of system parameters for optimum performance of control systems simulated on repetitive analog computers (see Section 3.3).
2. Solution of boundary value problems (see Sec. 3.8 for a typical example) by quickly varying parameters until the given boundary conditions are satisfied. This method can be applied to many vibration problems as well as to "eigenvalue" problems in atomic physics (solutions of Schrödinger's equation in quantum mechanics).
3. Rapid exploratory runs in order to conserve computing time on larger digital or analog computers, or in connection with manual computations.

Computing Elements for Repetitive Computers. Special Design Problems. Most of the computing elements used in repetitive d-c analog computers are essentially the same as those used in most other differential analyzers of the d-c analog type.

In the design of these components, due consideration must be given to the fact that the computing elements have to operate at frequencies of the order of 10,000 times higher than the computing elements of most "slow" d-c analog computers. Particular attention must be given to the high-frequency response of feedback amplifiers used in operational amplifier circuits (see Sec. 5.4) in order to avoid uncontrolled oscillations resulting from excessive phase shifts at high frequencies (see Secs. 4.2 and 4.5).

The theoretical requirement for high forward amplifier gain needed for achieving accurate integration is *not* essentially changed through the use of a "fast" time scale (see Sec. 4.3). Nevertheless, the amplifier forward gain used in repetitive computers at the present time is only of the order of 1,000 to 2,000, compared with d-c gains as high as 50,000 to 100 million used in "slow" d-c analog computers. This is chiefly due to the difficulty of maintaining high-frequency stability with such high gains and to the fact that to date no particular emphasis has been placed on the component accuracy of repetitive analog computers.

The higher frequencies used with repetitive computers also make the shielding of computing elements and interconnecting leads more critical than in the case of "slow" d-c analog computers.

The effects of drift and supply voltage variations on the accuracy of repetitive d-c analog computers vary with the computer design and must be considered in connection with the descriptions of specific examples of repetitive machines.

Whereas the linear computing elements such as integrators, summing amplifiers, and other operational amplifiers are basically capable of good performance at the high frequencies used in repetitive d-c analog computers, considerable difficulties are encountered in the design of function generators and multipliers for repetitive operation. The only applicable function generators seem to be circuits based on diode characteristics (see Sec. 6.7) and photoformers. Neither of these types of function generators is consistently capable of accuracies much better than 1 per cent, although some improvements in this respect may be expected. For multiplication, the crossed-fields electron-beam multiplier (see Sec. 6.1) has been successfully applied to a repetitive analyzer.[1] Other possible types of multipliers are quarter-square or logarithmic multipliers (see Sec. 6.1) based on function generators of the types mentioned above.

The Accuracy of Repetitive Analog Computers. Perhaps because of the relatively limited accuracy of the cathode-ray oscillographs used for viewing the solutions (see Sec. 7.3), repetitive analog computers have been regarded as useful chiefly for the quick, qualitative exploration of large numbers of solutions rather than for accurate computations. No repetitive analyzers with component accuracies better than about 1 per cent have been constructed. In reality, the limitations of the cathode-ray tube indicators as such need not stand in the way of developing repetitive analyzers capable of good accuracies, since it is possible to view the solutions on the cathode-ray tube screen and then to record selected solutions accurately by means of a photographic recorder of the electromagnetic or crystal oscillograph type (see Sec. 7.3). These recorders are capable of accuracies of better than 0.5 per cent. Because of the great convenience and speed of operation of repetitive analyzers, it is to be hoped that more accurate machines of this type will be developed. Such developments are predicated mainly on the design of more accurate function generators and multipliers capable of high-frequency operation.

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[1] MacNee, A. B., An Electronic Differential Analyzer, *Proc. IRE*, November, 1949.

Resetting Methods. The repetitive d-c analog computer must contain provisions for making the computer inoperative for a brief period between 30 and 100 times a second in order to permit the introduction of the correct initial conditions for all machine variables. The methods described in Sec. 7.1 for introducing initial conditions by charging all integrating capacitors to the desired values are applicable to repetitive computers if sufficiently quick-acting relays such as Western Electric mercury relays,[1] synchronous vibrators, or even rotating switches are available.

An alternative method is to return the output voltages of all integrators to zero between successive computer runs and to include provisions for adding to the output voltage of each integrator a step-function voltage whose value corresponds to the desired initial condition. This latter method has been used most frequently with repetitive computers. It is especially applicable to studies of damped vibrations where each machine variable has a tendency to return to zero during the course of the solution. In such cases, it may not even be necessary to discharge the integrating capacitors at the end of each computer run.

The Design of Repetitive D-c Analog Computers. The design problems encountered in developing repetitive d-c analog computers are best illustrated by a discussion of existing installations.

The Philbrick Computer. The only commercially available repetitive analog computer was developed by the George A. Philbrick Researches, Inc.[2]

The computing elements of the Philbrick analog computer are mounted in small utility boxes and can be set up in a rack and connected to a "central unit" cabinet which contains the power supplies as well as various signal generators and the control circuits.

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[1] Type D 168 479.

[2] Boston, Mass.

The following *linear computing elements* (operational amplifiers) are available:

1. Coefficient setting units
2. Summing amplifiers
3. Integrators
4. Differentiators
5. "Augmenting differentiators" [transfer function $(1 + aP)$]
6. "Augmenting integrators" $\left[\text{transfer function } \left(1 + \frac{1}{aP}\right)\right]$
7. "Unit lag" or "time-delay" units $\left[\text{transfer function } \left(\frac{1}{1 + aP}\right)\right]$

The following *nonlinear computing elements* are based on diode characteristics (see Sec. 6.7) and are especially suitable for the analysis of automatic control systems (see also Section 3.3 for a discussion of such circuits).

1. Limiters
2. Inert zone units
3. Hysteresis or backlash units

In addition to these computing elements, universal function generators also based on diode characteristics (see Sec. 6.7), quarter-square multipliers based on diode characteristics (see Sec. 6.1), and a number of computing elements simulating special control components have been introduced.

The voltage range used for the machine variables in the Philbrick analog computer is between plus and minus 50 volts. Overloads are indicated by means of neon bulbs. The nominal accuracy of all computing elements is between 1 and 2 per cent of full scale.

Figure 8.6 shows the block diagram of a typical Philbrick operational amplifier. The operational amplifier is of the conventional parallel-feedback type (see Sec. 4.6), but all input voltages are applied through cathode followers in order to obtain high input impedance and to eliminate possible interaction between computing elements. Every Philbrick computing element contains a phase inverter, so that both positive and negative output voltages can be obtained. Each Philbrick operational amplifier must be balanced in the usual manner (see Sec. 5.1) so as to show zero output voltage at each of its two output terminals for zero input voltage; d-c amplifier drift can introduce errors as in other d-c analog computers.

Introduction of Initial Conditions. The Philbrick analog computer will generate solutions periodically at a normal repetition rate of 60 cps. In many problems involving damped oscillations, all machine values return to the value zero in the course of the solution, so that all integrating capacitors will be discharged at the end of a computer "run." One can then introduce initial conditions by simply adding a step-function voltage equal in value to the desired initial condition to the corresponding machine variable at the beginning of each computer run. In the Philbrick analog computer, this is achieved by means of the ordinary summing amplifiers, the step functions being obtained from a square-wave generator in the central unit. If, on the other hand, one or more machine variables do not return to zero at the end of a solution, it is possible to short-circuit the integrating capacitors effectively at the end of each computer run by means of an electronic switch arrangement in the central unit. The discharging process may take as much time as a computer run. In this case, solutions are generated at a repetition rate of only 30 cps, alternate cycles being used for discharging the integrating capacitors.

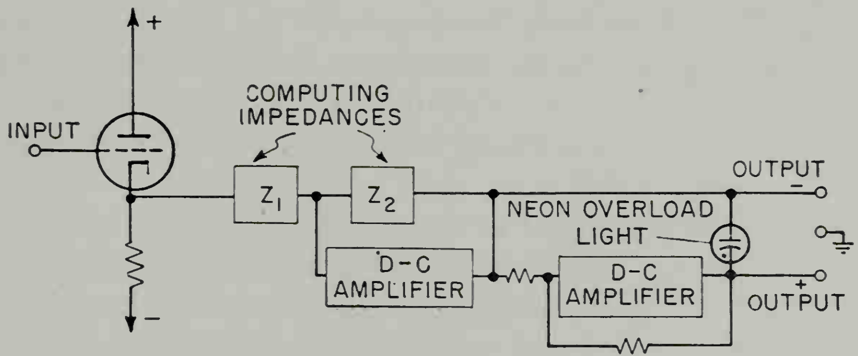

FIG. 8.6. Typical Philbrick operational amplifier.

Auxiliary Equipment. The central unit of the Philbrick analog computer contains plus and minus 300-volt power supplies, a square-wave generator for generating initial condition voltages and step voltages, a sine-wave generator for generating sinusoidal functions of the time, and a random noise generator. A sweep generator provides linear sweep voltages for the cathode-ray oscillographs used to display the solutions, and an electronic switch permits the display of two variables on one oscillograph screen. The electronic switch could also be used for certain switching applications during the computation.

Discussion of the Philbrick Analog Computer. In spite of its relatively low accuracy, the Philbrick analog computer can be an invaluable aid in engineering planning and design, especially in the development of automatic control systems.

Although the use of cathode-follower input circuits eliminates amplifier loading and interaction between computing elements, such arrangements also tend to decrease the computer accuracy. The Philbrick computer makes use of d-c amplifiers with their attendant difficulties of operation, although the repetitive solutions need not actually contain a d-c component. The newer MIT repetitive differential analyzer represents a significant improvement in this respect.

The MIT Repetitive Analyzer. A new repetitive differential analyzer was developed under the direction of Dr. A. B. MacNee[1] at the Research Laboratory of Electronics of the Massachusetts Institute of Technology.[2] This machine comprises parallel-feedback integrators and summing amplifiers as well as function generators of the photoformer type (see Sec. 6.5) and the crossed-fields electron-beam multiplier (see Sec. 6.1), newly developed by the same group. Initial conditions are again introduced by resetting all machine variables to zero at the end of each computer run and by adding step-function voltages corresponding to the initial conditions to each integrator output at the beginning of each computer run. The accuracy of the initial model is comparable to that of the Philbrick computer.

The significant improvement in the new analyzer is the use of a-c amplifiers in all operational amplifier circuits and other computing elements. Figure 8.7 shows a simplified diagram of an a-c amplifier used in the MIT computer. At the end of each computer run, all machine variables are reset to zero and all charges are removed from the coupling capacitors by means of electronic switches based on the clamping circuits commonly used in television sets.

At the end of each computer run, positive and negative *gating pulses* applied at the appropriate points shown in Fig. 8.7 permit the diodes to conduct and will reduce the output voltage of each amplifier stage (measured at the points x) to zero if the relative magnitudes of the positive and negative gating pulses are adjusted correctly by means of the balancing controls shown.

The use of these clamping circuits assures the return of the machine variables to the correct initial values for all types of solutions and also tends to simplify the computer design. Repetitive analyzers, such as the Philbrick computer, which make use of d-c amplifiers still require regulated power supplies in order to avoid drift in the reference level of the solutions. In repetitive analyzers using a-c amplifiers such as the MIT repetitive computer, power-supply regulation is not an essential requirement. Interaction between computing elements through the power supplies of repetitive computers can be eliminated by the use of conventional decoupling filters.[1] Furthermore, the a-c amplifiers used in the MacNee type of repetitive analyzer could be operated with a single power supply, although this is not the case in the original design. The component accuracy of the MacNee machine is between 0.5 and 1 per cent.

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[1] MacNee, *op. cit.*, p. 1315.

[2] Cambridge, Mass.

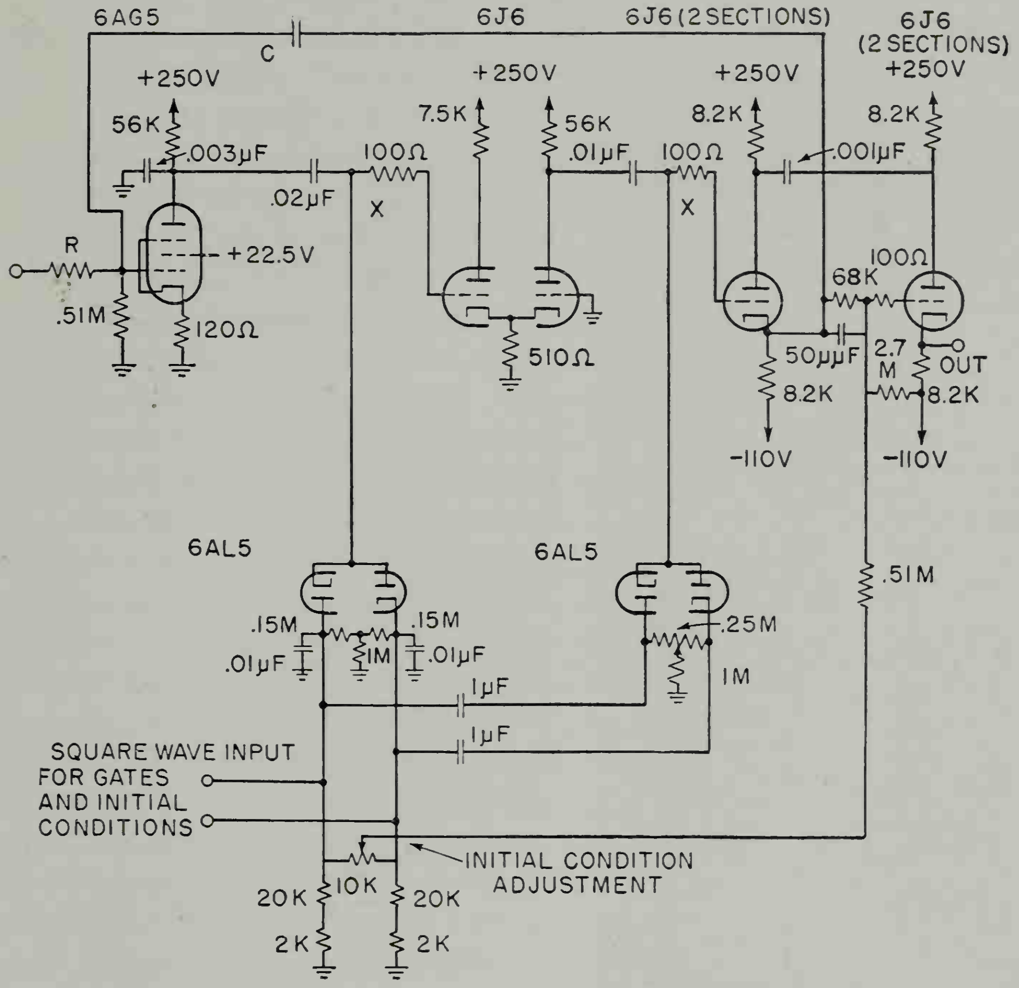

FIG. 8.7. Circuit diagram of an integrator from the MIT repetitive computer. Note the clamping circuits, used to discharge the integrating capacitor, and the summing circuit, used to add initial condition voltages to the integrator output voltage. The initial condition voltage is *not* used to charge the integrating capacitor directly in this arrangement.

8.6. Larger Computer Installations. *Introduction.* The following descriptions of some representative larger d-c analog computers built in the United States will serve to illustrate how different designers have attempted to deal with the design problems previously discussed.

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[1] See F. E. Terman, *Radio Engineers' Handbook,* McGraw-Hill, New York, 1943, for a discussion of decoupling filters.

The Goodyear Electronic Differential Analyzer (*GEDA*). The Goodyear Aircraft Electronic analog computer, originally developed by the Goodyear Aircraft Corporation[1] in connection with an Air Force contract, has been made available commercially in recent years.

The basic unit of the Goodyear Aircraft Electronic analog computer, shown in Fig. 8.8, is assembled complete in one movable cabinet. This unit comprises 20 high-quality d-c amplifiers, 4 double-diode parallel-type limiters, 2 double-diode series type limiters (similar to the "isolated" units described in Sec. 6.7), 10 coefficient setting potentiometers of the helical type, a patch bay for wiring problems and interconnecting computing elements, an accurate resistance-measuring bridge, and a number of relays used for setting up initial conditions. Also contained in the computer cabinet are an a-c voltage regulator which provides regulated a-c for the filaments of all but the power amplifier tubes, a small d-c power supply for relay operation, and three precision-regulated high-voltage power supplies. The latter will supply plus and minus 200 volts at 500 ma with an internal resistance of 0.05 ohm at zero frequency and minus 450 volts at 50 ma with 0.1-ohm internal impedance at zero frequency. These three high-voltage power supplies are regulated electronically by reference to one gas discharge tube used as a common reference element. The primary power for the operation of the Goodyear computer comes from the 110-volt, single-phase, 60-cycle a-c line. The average power consumption is 1,000 watts.

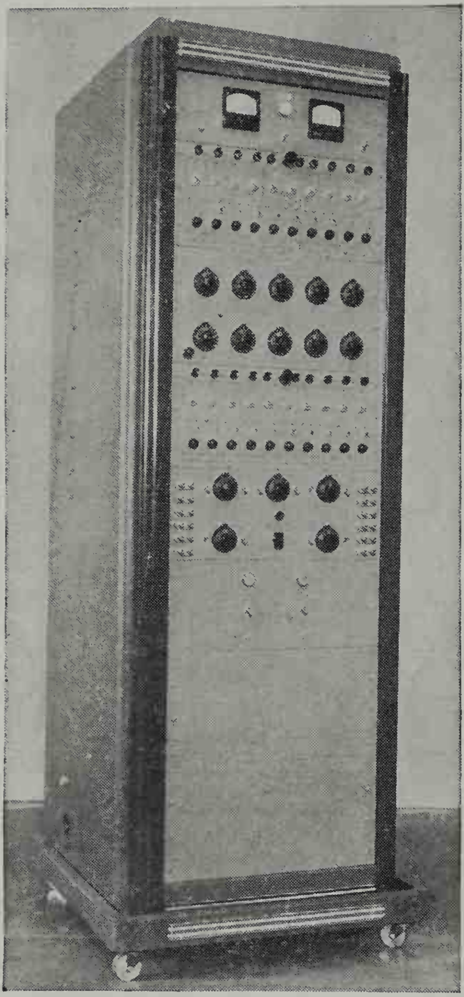

FIG. 8.8. Front view of the basic unit of the Goodyear Aircraft electronic analog computer.

The d-c amplifier design used in the Goodyear analog computer is a particularly ingenious one; the circuit used is shown in Fig. 5.35 and described in Sec. 5.5. Although the design of the Goodyear d-c amplifiers would permit the addition of automatic balancing units (see Sec. 5.6), the designers have not considered it necessary to introduce the added complexity and maintenance associated with such devices because of the very low drift of the excellent d-c amplifier circuit used. The input and output terminals of the d-c amplifier are brought out to jacks on the rear of two large chassis each bearing 10 individual plug-in d-c amplifier units. A central patch bay or jack board is conveniently arranged between two double rows of amplifier terminals. Various operational amplifiers can be built up by plugging combinations of resistors and capacitors into this patch bay and by connecting them to nearby amplifier terminals by means of patch cords. Each Goodyear d-c amplifier can be used in summing amplifier or summing integrator circuits with five or more input resistors, provided that the resistance of the input resistors in parallel is not less than 20,000 ohms. The patch bay also serves for interconnecting different computing elements by means of additional patch cords.

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[1] Akron, Ohio.

A resistance bridge incorporated in the computer makes it possible to select plug-in resistors to within 0.1 per cent of the desired value.

Ten polystyrene capacitors for use in integrators and other operational amplifiers are also supplied with the computer. A special push button permits the operator to discharge all these capacitors before each computer run so as to provide an initial-condition voltage equal to zero across each capacitor. Initial-condition voltages different from zero may be introduced into four integrators by means of four separate reset relays (see Sec. 7.1) connected to either the plus or minus 200-volt power supply through a voltage divider and four voltage-adjusting potentiometers. In order to obtain the correct voltage from the voltage divider, it is necessary to charge each integrating capacitor *in turn.* A step-function switch also permits the operator to introduce further initial conditions in the form of step voltages (see Sec. 7.1) and is generally useful in many computer applications.

The company will furnish a set of high-quality driver amplifiers (see Sec. 7.3) suitable for use with a direct-inking magnetic recorder as optional equipment. *The operating control system of the Goodyear computer* comprises a "compute" switch (here called the function switch) which actuates the reset relays and places the machine in the COMPUTE condition. In order to return the computer to the RESET condition after each computer run, it is necessary to

1. Release the compute switch
2. Discharge the integrating capacitors by pressing a "discharge" button
3. Charge up to four of the integrating capacitors with the respective initial-condition voltages by setting a selector switch and pressing a "charge" button

An elegant overload indication system of the common-bus type shown in Fig. 7.12*b* is used in this computer.

Multipliers and Function Generators. The *Goodyear nonlinear analog-computer unit* has been designed as an accessory to the basic Goodyear computer. This unit consists of a relay rack containing up to 12 computer servomechanisms as well as various associated amplifiers and power supplies. The frequency response of these servomechanisms is such that no noticeable phase shift is observed at 5 cps; errors are less than 5 per cent at 10 cps. Each servomechanism is of the modulated a-c type (60 cps) described in Sec. 6.4 and is capable of driving one to three potentiometers for multiplication and/or function generation. The function generating potentiometers supplied with this unit are of the universal tapped potentiometer type described in Sec. 6.6 and are equipped with interchangable plug-in networks using fixed resistors for establishing the correct tap voltages. The arrangement was originally intended to generate monotonic functions only, but any reasonable type of function could be generated with only minor circuit modifications.

All Goodyear computer units may be combined in groups to form larger computer installations. Electronic multipliers are under development.

Discussion of the Goodyear Aircraft Analog Computer. The outstanding feature of the Goodyear computer is the excellent electrical and mechanical design of the d-c amplifiers and of the compact power-supply system. The balancing and overload systems are also well designed. The construction of the various units is of good quality.

The use of plug-in summing and feedback resistors together with a calibrating bridge is not quite so convenient as the REAC system described later, but the Goodyear arrangement is more flexible as well as appreciably cheaper. The arrangement of the d-c amplifier terminals around the centrally located patch bay is practical and makes for short interconnections while yet permitting d-c amplifiers to be replaced without disturbing the computer setup. Note that the patch bay is *not* permanently connected to the d-c amplifiers as in the REAC.

The arrangement of the patch bay in the rear of the computer cabinet may conserve panel space and be desirable in applications which require few changes in the computer setup. For all-round experimental purposes, however, the necessity of making setup changes in the rear of the computer cabinet may have a definite nuisance value.

The component accuracy claimed for the Goodyear machine is 0.1 per cent for linear computing elements and less than 0.5 per cent for most nonlinear operations.

The weakest point in the design of the Goodyear computer is its operating control system; as many as four switching operations may be necessary to return the machine to the RESET condition after each computer run. The complication of the resetting operation is the high price paid for the omission of a low-impedance reference voltage supply in the main computer unit.[1]

The Reeves Electronic Analog Computer (REAC). The REAC was designed by the Reeves Instrument Corporation[2] under a contract with the United States Navy. The REAC has been made available commercially and is probably the most frequently used of all d-c analog computers.

The *basic REAC unit* comprises a relay rack containing 20 high-quality operational amplifiers, four double-diode limiters, and a number of coefficient setting potentiometers of the helical type. A separate power-supply rack houses plus 350-volt, minus 190-volt, and minus 350-volt regulated power supplies for the operation of the d-c amplifiers as well as plus and minus 100-volt regulated power supplies for computing voltages, a regulated d-c filament supply for the input tube filaments, a d-c supply for relay operation, and various filament transformers. All high-voltage power supplies are regulated by reference to one dry battery used as a common reference element.

The REAC d-c amplifiers are of the self-balancing type described in detail in Sec. 5.6. Booster resistors (see Sec. 5.6) may be used to increase each amplifier power output when necessary. Seven of the twenty d-c amplifiers in the basic REAC unit are merely used as phase inverters with a single input terminal and a gain of -1. There are also six summing amplifiers and seven summing integrators, each having seven input terminals corresponding to the multiplying coefficients -1, -1, -1, -1, -4, -4, and -10 as illustrated in Fig. 2.1. The summing and feedback resistors are selected so that each resistance is within 0.1 per cent of the specified value, and the capacitances of the integrating capacitors (plastic dielectric type) are accurate to within 0.25 per cent.

An optional separate *servomechanisms unit* comprises a relay rack containing four computer servomechanisms and a separate power-supply rack. The computer servomechanisms utilize 60-cps induction motors driven by a-c amplifiers (see Sec. 6.4); d-c input voltages are converted to alternating current by means of synchronous converters. Each Reeves computer servomechanism is arranged so that it can perform the following operations:

1. Multiplication of a machine variable (d-c voltage) by one to three other machine variables

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[1] Recent models of the GEDA do have a reference power supply, but to the best of the authors' knowledge, the operating control system has not been changed.

[2] New York, N.Y.

2. Multiplication of a machine variable by one to three functions of other machine variables by means of optional cylindrical-type function potentiometers (see Sec. 6.5)
3. One transformation from Cartesian to polar coordinates or inversely (use as a *vector servo*, as in Sec. 6.8)

The frequency response of the Reeves servomechanisms is flat (0.5 per cent accuracy) up to about 5 cps.

REAC Construction. All input and output terminals of the REAC computing elements are brought out to jacks on large central *patch bays*. The wiring used for making these connections is not shielded, so that there may be noticeable cross talk at signal frequencies higher than 2 cps. The REAC possesses a *push-button power control system* similar to the one illustrated in Fig. 7.8. The *operating control system* is similar to the simple switch-type operating control system shown on the right of Fig. 7.5 and is physically separated from the power control system.

A complete REAC installation, comprising a basic unit, a servomechanisms unit, and two power-supply racks, is shown in Fig. 1.15. Several REAC units may be combined to form larger computer installations. Several types of *recorders* (see Sec. 7.3 and Figs. 7.20 and 7.21) may be furnished with the REAC as optional equipment. The company will also furnish special driver amplifiers (see Sec. 7.3) for recorder operation.

For best accuracy, the time scales used with the REAC should be chosen so that the signal frequencies are between 0.02 and 5 cps, but both faster and slower computations are possible.

Discussion of the REAC: Computing Elements. The operational amplifiers used in the REAC are capable of accurate performance (better than 0.5 per cent integrator accuracy for a computing time of 2 min in most cases). This is a result of the utilization of the RCA automatic balancing system (Sec. 5.6) in connection with carefully chosen input and feedback network components.

The other computing elements are of comparable quality, although the frequency response of the computer servomechanisms leaves something to be desired. The provision for vector resolution in each REAC computer servomechanism is useful for a number of applications but necessarily raises the cost of the servomechanisms unit rather substantially.

Discussion of the REAC: Composite Design. The mechanical construction and wiring of REAC units are of excellent quality, although perhaps unnecessarily expensive.

The REAC operating control system is very simple and easy to maintain, if not impressively convenient for the operator. The compute-reset-hold switch is almost hidden on the front panel, and the recorder paper feed cannot be started and stopped from the control panel. No remote-control station is furnished with the REAC, but such a device could be added easily. "Plugboard" type patch bays for multishift operation (see Sec. 8.3) have been made available with some late model REAC's; such a patch bay is shown in Fig. 8.9. Finally, jack panels containing resistors and capacitors for making up special computing networks (see Sec. 4.6) are available with the late-model machines.

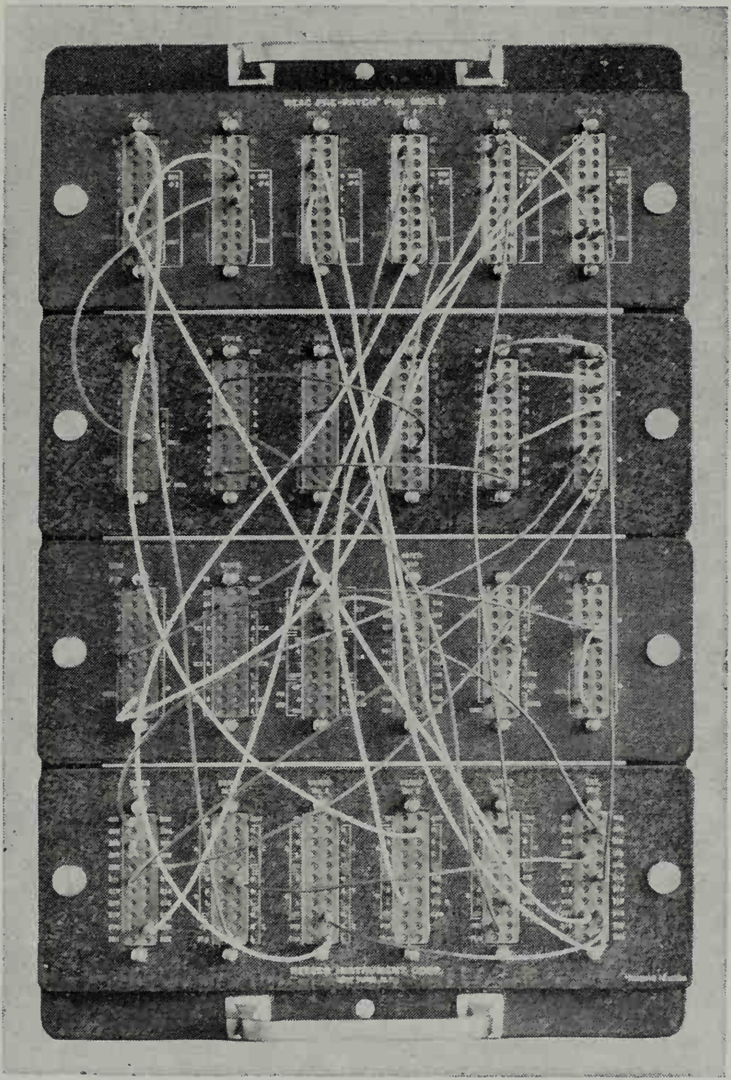

FIG. 8.9. Removable plugboards used for setting up problems on the newer REAC computers.

8.7. Development Trends in the Design of Larger D-c Analog Computers. *Introduction.* The development of the operational amplifiers used for performing linear operations in d-c analog computers has reached a high state of perfection. The principal need for further development is in the field of multipliers and function generators capable of accurate operation at signal frequencies between 0 and 5,000 cps (see Chap. 6).

Until these developments can be completed, the chief improvements on existing d-c analog-computer designs can be made with respect to *setup and operating convenience.* The following descriptions of modern computing equipment will show examples of such improvements.

The Curtiss-Wright Analog Computer (*CURTIAC*). The CURTIAC, built under a contract with the United States Navy Department,[1] represents a joint effort of the Curtiss-Wright Corporation, Airplane Division,[2] and the Applied Physics Laboratory, Johns Hopkins University.[3]

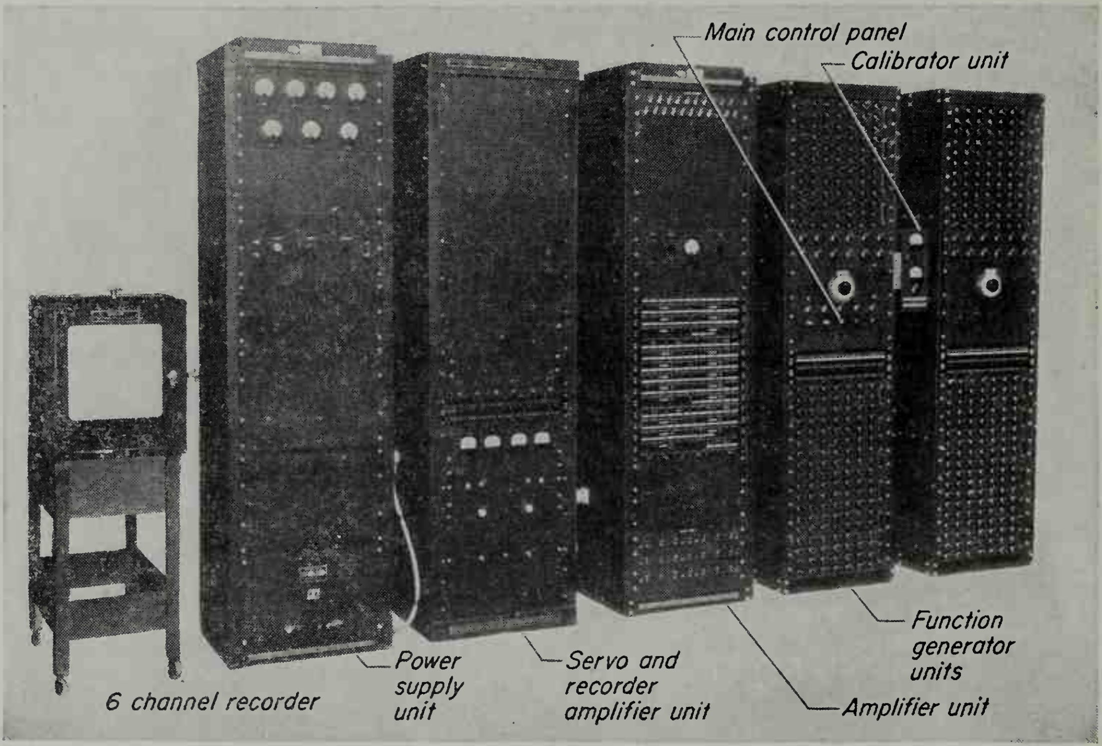

FIG. 8.10. The Curtiss-Wright analog computer (CURTIAC).

Figure 8.10 shows the complete CURTIAC installation. The power-supply unit (first from the left) contains a push-button power control system like the one used in the REAC, seven meters indicating power-supply output voltages and currents, and several power supplies, including two reference power supplies, all essentially similar to those used in the REAC. The amplifier unit (third from the left) contains 20 self-balancing computing amplifiers with all terminal connections brought out to a large patch bay on the front panel. The high-quality operational amplifiers used in the CURTIAC are designed essentially like the amplifiers used in the REAC, with the exception of some minor circuit refinements. Unlike in the REAC, each operational amplifier can be used as either a summing amplifier or a summing integrator with seven input terminals corresponding to the multiplying factors −1, −1, −1, −1, −4, −4, and −10. Each amplifier can be changed over from summing amplifier to summing integrator operation by means of a switch located in the rear of each amplifier chassis.

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[1] Clearance of this material by the United States Navy Department should not be construed to imply that the opinions expressed here are those of the Navy Department.

[2] Columbus, Ohio.

[3] Silver Spring, Md.

The second relay rack unit from the left in Fig. 8.10 contains six simple recorder driver amplifiers using the low-cost circuit shown in Fig. 7.16. The calibrating controls can be seen in the center of the front panel. This rack also contains a number of multiplying or function generating servomechanisms and the associated power supplies. Each computer servomechanism is of the modulated a-c type; the servo amplifiers were constructed by modifying standard public-address amplifiers, with excellent results. Provisions are made for operating the servomotors at about three times their rated power for the short time of each computer run. This arrangement results in a considerably improved torque-to-inertia ratio (see Sec. 6.4). The computer servomechanisms used in the CURTIAC have a frequency response flat up to about 5 cps.

Function Generator Units. Universal function generating potentiometers of the type discussed in Sec. 6.6 may be utilized in the CURTIAC in connection with the computer servomechanisms. Twenty additional universal function generating potentiometers driven by a large synchronous motor are used to generate functions of the machine time. These function generating potentiometers are arranged on shafts running vertically through each of the two function generating racks shown on the right of Fig. 8.10. The function adjusting networks for each of the 20 universal function generating potentiometers are of the plug-in type. They are mounted on interchangeable relay rack panels each of which is provided with 21 controls for setting the various tap voltages. Figures 6.43*b* and 6.43*c* show a photograph of an individual unit taken from the CURTIAC.

Each CURTIAC function generator unit also contains 12 coefficient setting potentiometers of the helical type and a patch bay for making connections to the various computing elements. The shafts driving the potentiometers in the two function generator units are coupled together by a solid horizontal shaft geared down from the main shaft, although a synchro transmission is also provided for remote operation of one of the function generator units. The elapsed-time indicators seen on the front panel of each of the function generator units indicate the computing time elapsed as determined by the displacement of the potentiometer shaft.

Also located in one of the function generator units are four double-diode-type limiters and the computer operating control panel.

CURTIAC Control System. The operating control system in the CURTIAC is perhaps the most interesting feature of this computer. This is a *push-button control system with automatic timing* and uses a circuit similar to the one shown in Fig. 7.10 and discussed in Sec. 7.2. All push buttons are duplicated on a portable *remote-control station* which can be plugged into the control panel. Figure 8.11 shows the portable remote-control station as well as the calibrator unit described further below.

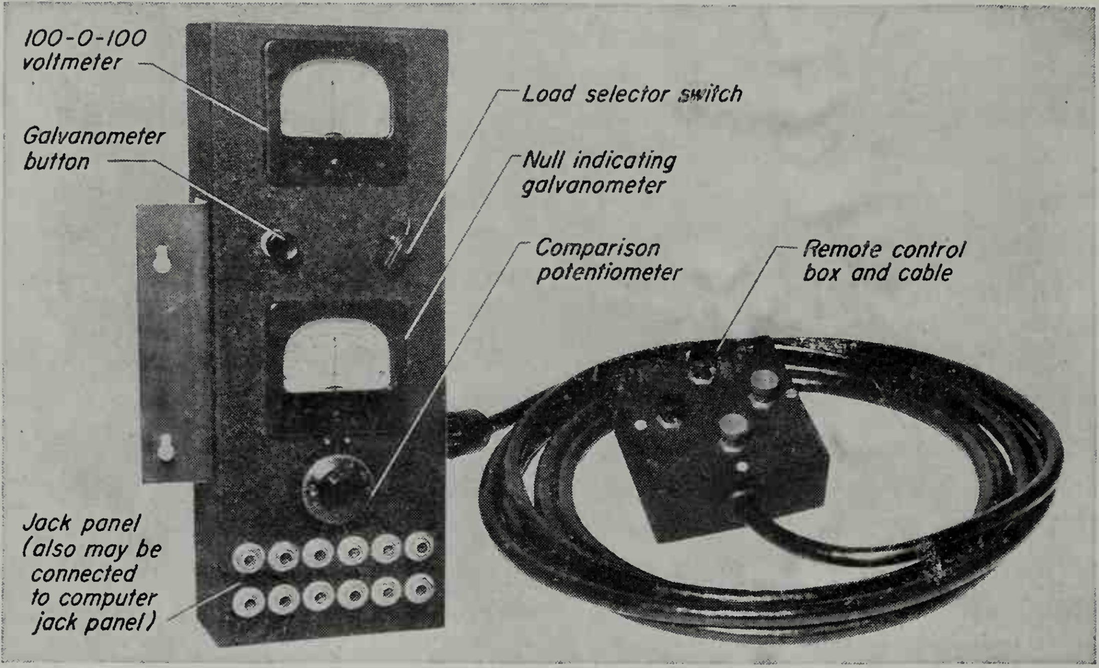
100-0-100 voltmeter
Galvanometer button
Load selector switch
Null indicating galvanometer
Comparison potentiometer
Remote control box and cable
Jack panel (also may be connected to computer jack panel)

FIG. 8.11. Curtiss-Wright CURTIAC calibrator unit and remote-control station.

Calibrator Unit. The calibrator unit furnished with the CURTIAC is a simple but versatile measuring instrument housed in a steel utility box which can be conveniently attached to any of the computer relay racks as shown in Fig. 8.9. The circuit of the CURTIAC calibrator is shown in Fig. 8.12; the unit contains a voltmeter to be used for rough voltage measurements and an accurate helical-type potentiometer with a microammeter null indicator. The latter components may be used for accurate voltage measurements in conjunction with the computer reference power supply. They may also be used in a bridge circuit for measuring resistance. The calibrator is specifically intended as a comparison potentiometer for setting the tap voltages of the CURTIAC universal function generating potentiometers. For this reason a *load selector switch* permits one to connect substitute load resistances, equal to the input resistances of the CURTIAC operational amplifiers, to the function potentiometer being calibrated. The calibrator unit is also useful for setting loaded constant coefficient potentiometers, for accurate amplifier gain measurements, and as a measuring instrument of general utility.

The synchronous motor driving the time shaft in the CURTIAC may be subject to speed changes due to variations in the power line frequency (see Sec. 6.5). This difficulty could be avoided by supplying the time motor with alternating current from a power amplifier driven by a tuning-fork oscillator (see Secs. 6.5 and 7.3).

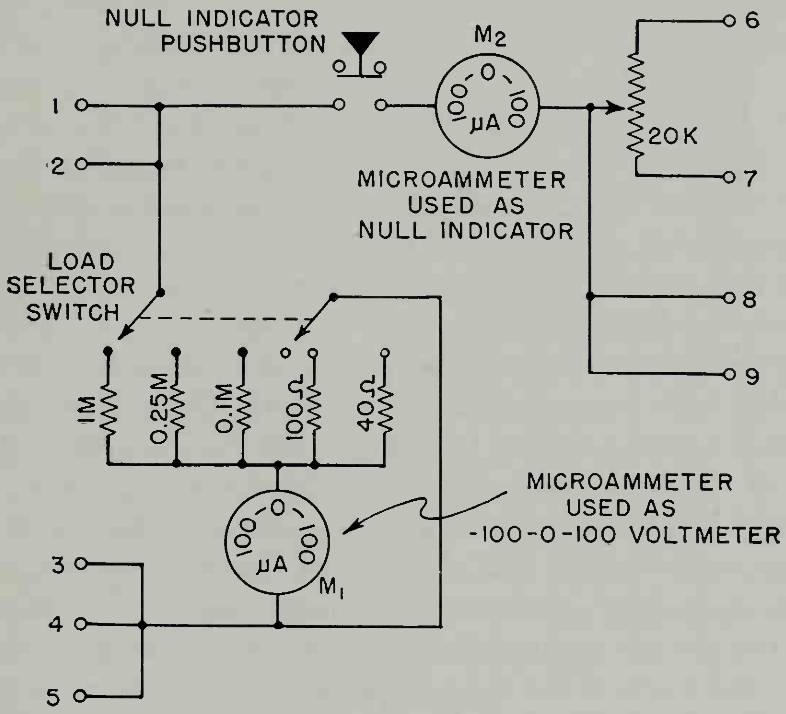
NULL INDICATOR PUSHBUTTON
M2
0
100
100
μA
20K
1
2
3
4
5
6
7
8
9
MICROAMMETER USED AS NULL INDICATOR
LOAD SELECTOR SWITCH
1M
0.25M
0.1M
100 Ω
40 Ω
MICROAMMETER USED AS -100-0-100 VOLTMETER
M1

Fig. 8.12. Calibrator unit for a d-c analog computer (CURTIAC). The numbered points are brought out to jacks on the calibrator unit and, if desired, through a special cable, also to jacks in the main computer patch bay. The *load selector switch* changes the voltmeter resistance so as to simulate different loads on the computing element being calibrated. The voltmeter scale does *not* change.

The RAND Analog-computer Installation. The analog-computer installation of the RAND Corporation[1] is particularly interesting, since it was developed for the often involved and sophisticated requirements of a large computer laboratory which also comprises much digital computing equipment. Although the RAND analog computer is based on simple modifications of two early-model REAC's, a number of successive improvements have resulted in a machine whose accuracy and usefulness are vastly superior to the original equipment. Eventually, a component accuracy better than 0.05 per cent is hoped for.

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[1] Santa Monica, Calif. The machine was developed for the company's own use only.

Since the RAND analog computer is not used for partial systems tests which require operation on a 1:1 time scale, the machine is usually operated on relatively slow time scales which permit optimum accuracy in the computer servomechanisms used. Accordingly, provisions have been made to utilize larger values of integrating capacitances and integrator summing resistances.

The original Reeves d-c amplifiers have been modified; each RAND d-c amplifier can be used as a general-purpose operational amplifier as well as for summing and integrating. Each amplifier has been provided with the well-designed automatic balancing circuit discussed in Sec. 5.6 and shown in Fig. 5.39. The d-c amplifier circuits comprise cathode-follower input stages having very low grid currents (10^{-6} μa) and error-indicator lights (see Sec. 5.6). The use of grid-blocking capacitors was decided against because of their poor overload performance.

The original Reeves computer servomechanisms have been redesigned and provided with input summing networks. Correction function generators (see Sec. 6.5) are used to improve the accuracy of multiplication. A number of computing relays (see Sec. 6.5) have been added.

The relatively slow time scales used permit the utilization of Esterline-Angus magnetic recorders and of modified Reeves servo drum recorders for accurate recording. Several of the Reeves servo drum recorders have been modified for use as universal function generators or input tables in the manner discussed in Sec. 6.5 and permit the generation of any reasonable functions of machine variables.

A digital readout device, or voltmeter reading in digits, is provided, and the installation of an electric typewriter for recording the output in tabular form is planned.

The power supplies in the original computer are supplemented by a new precision reference power supply which comprises an automatic balancing circuit.

Setup and Control System. The outstanding feature of the RAND analog computer is the setup and control system. Since the RAND computer is intended for possible three-shift operation, all the terminals of the d-c amplifiers, feedback networks, potentiometers, relays, servomechanisms, input tables, and recorders are brought out to an IBM-type patch panel (see Sec. 8.3). Interconnections for each computer setup are wired up on individual plugboards which are then plugged into the patch panel. Each plugboard can be prepared independently while the machine is being used for a different problem. After completion of wiring, the

TABLE 8.1. COMPARATIVE REFERENCE DATA ON FOUR REPRESENTATIVE D-C ANALOG COMPUTERS

	Boeing	Goodyear	Reeves	Curtiss-Wright
Number and type of d-c amplifiers in basic unit[1]	12 (Regenerative; circuit of Fig. 5.32)	20 (Circuit of Fig. 5.35)	20 (Similar to circuit of Fig. 5.38*a*)	20 (Similar to circuit of Fig. 5.38*a*)
Automatic balancing..........	No (could be added)	No (could be added)	Yes	Yes
Input and feedback networks...	Plug-in; all amplifiers have reset relays	Plug-in; only 4 initial-condition relays	7 Phase inverters, 6 summing amplifiers, 7 summing integrators	Switch permits each amplifier to be used as summing amplifier or summing integrator. Inputs: 4 × gain 1, 2 × gain 4, and 1 × gain 10
Coefficient setting potentiometers	12 helical type	10 helical type	20 helical type	24 helical type
Limiters (each unit limits both positively and negatively)	Parallel-diode type; equal plus and minus limits; isolated type	4 parallel-diode type with cathode followers; 4 series-diode type	4 parallel-diode type; amplifier type optional	4 parallel-diode type
Number and type of multipliers in basic multiplier unit[1]......	None available at present	Servo type (up to 12)	4 servo type	4 servo type (see Sec. 6.4)
Function generators...........	None available at present	Tapped potentiometers driven by servos (up to 12)	Tapered potentiometers driven by servos (see Sec. 6.5)	Tapped potentiometers (see Sec. 6.6) driven by time motor or servos
High-voltage power supplies[2] in basic unit[1]................	+300 volts, −300 volts; battery reference supply	+200 volts, −200 volts, −250 volts	+300 volts, −350 volts, −190 volts; +100 volts, −100 volts reference supplies	+300 volts, −350 volts, −190 volts; +100 volts, −100 volts reference supplies
Low-voltage power supplies[2] in basic unit[1]................	A-c filament operation (regulated)	Regulated a-c and unregulated a-c filament operation; D-c for relays	Regulated d-c; also a-c; d-c for relays	Regulated d-c; also a-c; d-c for relays
Operating controls............	Switch	Switch plus charging and discharging push buttons	Switch	Push button with automatic timing (similar to circuit of Fig. 7.10)
Repetitive operation..........	Yes, if desired	No	No	No
Calibrator....................	Comparison potentiometer, if desired	Special resistance-measuring bridge	Use servo	Special unit or use servo
Vector servos.................	No	Must be set up	Yes	Must be set up
Hold condition................	No	No	Yes	Yes

[1] Several basic units may be combined in each case to form larger computer installations.
[2] All computers described may be operated from the 110-volt, 60-cycle a-c line.

interconnections on a plugboard may be checked from the reverse side of the board through the use of a buzzer or ohmmeter. If improper behavior of the computer is suspected, a specially wired "diagnostic" or test problem plugboard can be plugged into the machine. If mathematical difficulties arise, or if subsequent parameter choices depend on the analysis of results to date, the problem can be removed immediately and another one started. Plugboards can be filed for later use.

On changing problems it is, of course, necessary to change the functions on the input tables and to reset the coefficient setting potentiometers. In order to permit the greatest possible operating convenience, the patch panel and all coefficient setting potentiometers have been combined with the computer operating controls in a large control console; all controls can be reached by the operator from a sitting position. A single push button will connect a test voltage source across the input terminals of all coefficient setting potentiometers; their settings can then be read directly, *with the load connected*, on a servo-driven comparison potentiometer which may be switched into each circuit. A push-button operating control system is used.

Construction. Physically, the d-c amplifiers, servomechanisms, and power supplies are located in enclosed cabinets whereas the feedback networks are located inside the control console, where they are not affected by the heat generated in the amplifiers and power supplies. The various units are interconnected with well-insulated cables; no cross-talk difficulties have been observed at the low computing speeds used. The patch panel and plugboards are made of polystyrene in order to minimize leakage.

Discussion of the RAND Computer. The RAND analog computer is a unique and imaginative development whose potential utility is equal to that of much larger machines. Computing speed has been successfully traded for increased accuracy in multiplication, function generation, and recording. It will be interesting to see to what extent multishift operation of the type planned for the RAND computer will pay for the relatively high cost of the installation.

Some Possible Design Features for a Future Improved D-c Analog Computer. A future improved d-c analog computer might utilize push-pull operational amplifiers. This would

1. Permit a slightly improved computing accuracy.
2. Permit the utilization of only two high-voltage power supplies for d-c amplifier operation.
3. Result in a more constant load on these power supplies.

4. Eliminate a number of phase-inverting amplifiers in most computer setups. On the other hand, the number of feedback and input networks and of coefficient setting potentiometers would be doubled.

As an example, Fig. 8.13 shows the block diagram of a push-pull integrator.

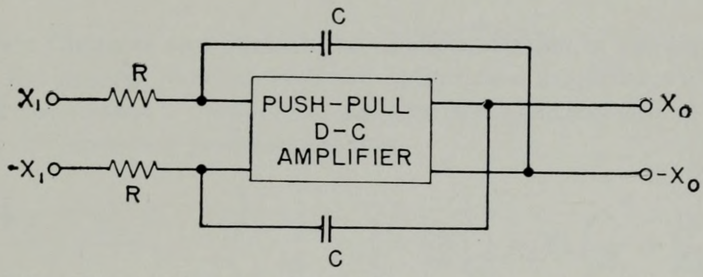

FIG. 8.13. Block diagram of a self-balancing push-pull integrator.

APPENDIX: SOME PROPERTIES OF PARALLEL-FEEDBACK-TYPE OPERATIONAL AMPLIFIERS

Effects of Gain Changes and Distortion. Differentiation of the performance equation (4.69) of the general parallel-feedback operational amplifier with respect to the forward gain A shows that *a change of* ϵ_A *per cent in the forward gain* A *of the d-c amplifier will cause a corresponding percentage change*

$$\left|\epsilon_A \frac{1 + Z_o\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)}{(1 - A) + Z_o\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)}\right| \qquad \left\{\approx \left|\frac{\epsilon_A}{A}\left[1 + Z_o\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)\right]\right| \right.$$
$$\left. \text{if } |A| \gg 1 \text{ and } \left|\frac{Z_o}{1 - A}\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)\right| \ll 1\right\} \qquad \text{(A.1)}$$

in the output voltage X_o *of the operational amplifier of Fig.* 4.24*b* (the simpler circuit of Fig. 4.24*a* is a special case of the circuit of Fig. 4.24*b*).

Again, since distortion in the d-c amplifier may be considered in terms of gain changes as in Secs. 4.2 and 4.5, ϵ_D *per cent distortion in the d-c amplifier will cause a distortion percentage of*

$$\left|\epsilon_D \frac{1 + Z_o\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)}{(1 - A) + Z_o\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)}\right| \qquad \left\{\approx \left|\frac{\epsilon_D}{A}\left[1 + Z_o\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)\right]\right| \right.$$
$$\left. \text{if } |A| \gg 1 \text{ and } \left|\frac{Z_o}{1 - A}\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)\right| \ll 1\right\} \qquad \text{(A.2)}$$

in the output voltage of the operational amplifier of Fig. 4.24*b*.

Effects of Load Impedance and D-c Amplifier Output Impedance on Operational Amplifier Performance. The operational amplifiers discussed in the preceding sections have been analyzed without regard to the effects of other computing elements to which they will be connected. At times, it will be useful to consider the effects of the impedance of the load and the d-c amplifier output impedance on the performance of an operational amplifier. Figure A.1 shows the equivalent circuit of an operational amplifier whose d-c amplifier has the output impedance Z_I; the operational amplifier is connected to a load of impedance Z_L.

The performance equation of the circuit of Fig. A.1 is

$$X_o = Z_o \sum_i \frac{X_i}{Z_o} \frac{A'}{-A' + \left(Z_o \sum_i \frac{1}{Z_i} + 1\right)} \qquad \text{(A.3)}$$

with

$$A' = \frac{\left(\frac{1}{Z_o} + \frac{A}{Z_I}\right)}{\left(\frac{1}{Z_I} + \frac{1}{Z_o} + \frac{1}{Z_L}\right)} \qquad \text{(A.4)}$$

A comparison of the performance equation (A.3) with the simplified performance equation (4.68) shows that, for large values of $|A|$, the effect of the internal resistance Z_I of the operational amplifier is essentially to replace the forward gain A in Eq. (4.68) by the "equivalent forward gain" A' given by Eq. (A.4).

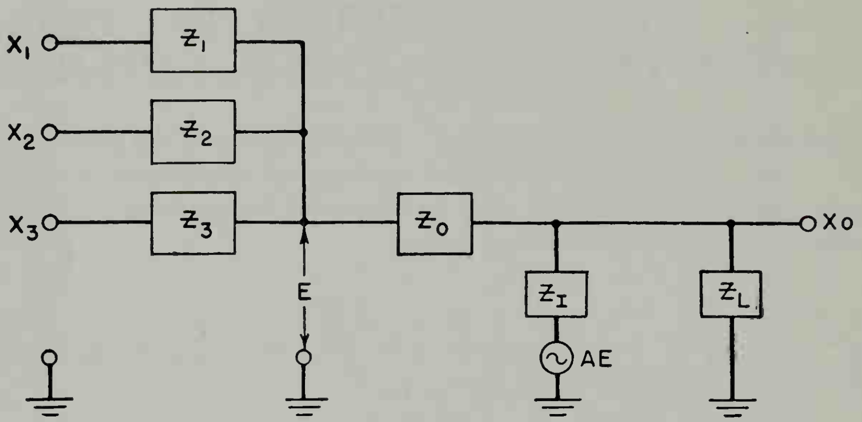

FIG. A.1. Equivalent circuit of an operational amplifier connected to a load.

For large values of the absolute value $|A|$ of the forward gain A,* the *output impedance* Z_S of the operational amplifier as such is

$$Z_S = -\frac{Z_I}{A}\left(Z_o \sum_i \frac{1}{Z_i} + 1\right) \tag{A.5}$$

In particular, for the case of the *summing amplifier* (Fig. 1.7*d*)

$$Z_S = -\frac{Z_I}{A}\left(R_o \sum_i \frac{1}{R_i} + 1\right) \qquad (|A| \gg 1) \tag{A.6}$$

For the *parallel-feedback integrator* (Fig. 4.18)

$$Z_S = -\frac{Z_I}{A}\left(\frac{1}{RCP} + 1\right) \qquad (|A| \gg 1 \qquad R_L \gg R) \tag{A.7}$$

and for the *summing integrator* (Fig. 4.20)

$$Z_S = -\frac{Z_I}{A}\left(\frac{1}{CP} \sum_i \frac{1}{R_i} + 1\right) \qquad (|A| \gg 1 \qquad R_L \gg R) \tag{A.8}$$

The *input impedance* seen by the *i*th input voltage source of the parallel-feedback operational amplifier of Fig. 4.24*b* is practically equal to the impedance Z_i if the quantity $\left|\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\right)\frac{Z_o}{1 - A}\right|$ is small, as the feedback tends to make the voltage X_0/A at the d-c amplifier input practically equal to zero.

In most applications, the output of each operational amplifier will be connected to one of the input terminals of another operational amplifier. The load resistance Z_L for the first operational amplifier will then be approximately equal to one of the input impedances Z_i of the second amplifier. Since the value of Z_I is usually of the order of

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* Chance, B., F. C. Williams, V. Hughes, D. Sayre, and E. F. MacNichol, *Waveforms*, MIT Radiation Laboratory Series, Vol. 19, McGraw-Hill, New York, 1949.

10,000 ohms, with the Z_i of the order of 1 megohm, the effects of Z_I and Z_L as given by Eqs. (A.3) and (A.4) are probably small. *Because of the complex feedback paths involved in complete computers, however, it may be difficult to find general expressions for the actual effective source and load impedances of each operational amplifier.* In order to gain more information, it may become necessary to solve the very complex network equations for the complete computer.

In actual practice, the output impedance of one operational amplifier is usually neglected as compared with the input impedance of the following one. The transfer function of a number of cascaded operational amplifiers may then be computed by simply multiplying their individual transfer functions.

BIBLIOGRAPHY

Artzt, M., Survey of D-c Amplifiers, *Electronics*, August, 1945.

Beer, A. C., and H. W. Boehmer, Cornell thesis on pulsed attenuators, under the direction of H. S. Sack.

Berkeley, E. C., *Giant Brains*, Wiley, New York, 1949.

Biberbach, L., *Differentialgleichungen*, Springer, Berlin, 1930.

Blackburn, J. F., *Components Handbook*, MIT Radiation Laboratory Series, Vol. 17, McGraw-Hill, New York, 1949.

Bode, H. W., *Network Analysis and Feedback Amplifiers*, Van Nostrand, New York, 1945.

Caldwell, S. H., and V. Bush, A New Type of Differential Analyzer, *J. Franklin Inst.*, **240**: 255, October, 1945.

Chance, B., F. C. Williams, V. Hughes, D. Sayre, and E. F. MacNichol, *Waveforms*, MIT Radiation Laboratory Series, Vol. 19, McGraw-Hill, New York, 1949.

Deeley, E. M., and D. M. MacKay, Multiplication and Division by Electronic Analogue Methods, *Nature*, **163** (No. 4147): 650, Apr. 23, 1949.

Den Hartog, T. P., *Mechanical Vibrations*, 3d ed., McGraw-Hill, New York, 1947.

Department of Electrical Engineering Staff, Massachusetts Institute of Technology, *Applied Electronics*, The Technology Press, Wiley, New York, 1943.

Dole, F. E., and L. A. Nettleton, Reducing Potentiometer Loading Error, *Rev. Sci. Instruments*, **18**: 332, 1947.

Edwards, C. M., and E. C. Johnson, An Electronic Simulator for Non-Linear Servomechanisms, *Trans. AIEE*, **69**, Paper 50-47, 1950.

Fink, D. G., *Engineering Electronics*, McGraw-Hill, New York, 1938.

Frayne, J. G., and H. Wolfe, *Elements of Sound Recording*, Wiley, New York, 1949.

Frost, S., Compact Analog Computer, *Electronics*, **21**: 116–122, July, 1948.

Ginzton, E. L., D-c Amplifier Design Techniques, *Electronics*, March, 1944.

Goelz, G. W., and J. F. Calvert, Analog Computer for Multi-Component Fractionation Calculations, *Trans. AIEE*, **69**, Paper 50-15, 1950.

Goldberg, E. A., Stabilization of D-c Amplifiers, *RCA Rev.*, **11**: 296, 1950.

Goodyear Aircraft Company, Aerophysics Department, *Operating and Maintenance Manuals for the Goodyear Electronic Analog Computer*, Akron, Ohio.

Greenwood, I. A., J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948.

Hagelbarger, D. W., C. E. Howe, and R. M. Howe, Investigation of the Utility of an Electronic Analog Computer in Engineering Problems, UMM-28, Aeronautical Research Center, Engineering Research Institute, University of Michigan, Ann Arbor, Mich., Apr. 1, 1949.

Harder, E. L., and G. D. McCann, A Large Scale General Purpose Electronic Analog Computer, *Trans. AIEE* (T 8112), **67**, 1948.

James, H. M., N. B. Nichols, and R. S. Phillips, *Theory of Servomechanisms*, MIT Radiation Laboratory Series, Vol. 25, McGraw-Hill, New York, 1947.

Koehler, J. S., An Electronic Differential Analyzer, *J. Applied Phys.*, **19**: 148–155, February, 1948.

Korn, G. A., Elements of D-c Analog Computers, *Electronics*, April, 1948.

———, Design of D-c Electronic Integrators, *Electronics*, May, 1948.

———, The General Difference Analyzer, *Curtiss-Wright Rept.* P 537-V-46, 1948.

———, Application of a New Analog Computer to the Integration of Euler's Equations for a Spinning Body, *Boeing Aircraft Company Document* D-9969, June 3, 1949.

———, Stabilization of Simultaneous Equation Solvers, *Proc. IRE,* **37**: 1000, 1949.

———, Design and Construction of Universal Function Generating Potentiometers, *Rev. Sci. Instruments,* **21** (No. 1): 77–81, January, 1950.

——— and T. M. Korn, Bridge-balanced D-c Amplifier, *Radio and Television News,* Engineering Edition, January, 1950.

———and T. M. Korn, Modern Servomechanisms Testers, *Elec. Eng.*, September, 1950.

Lauer, H., R. Lesnick, and L. E. Matson, *Servomechanism Fundamentals,* McGraw-Hill, New York, 1947.

Lovell, C. A., Continuous Electrical Computation, *Bell Lab. Record,* **25** (No. 3): 114–118, March, 1947.

MacKay, D. M., A High-speed Electronic Function Generator, *Nature,* **159** (No. 4038): 406–407, Mar. 22, 1947.

MacNee, A. B., A High-speed Product Integrator, *MIT RLE Rept.* 136, Aug. 17, 1949.

———, An Electronic Differential Analyzer, *Proc. IRE,* **37** (No. 11): 1315–1324, November, 1949.

Margenau, H., and G. M. Murphy, *The Mathematics of Physics and Chemistry,* Van Nostrand, New York, 1946.

McCann, G., The California Institute of Technology Electric Analog Computer, *Mathematical Tables and Other Aids to Computation,* Vol. III, No. 28, October, 1949.

———, S. W. Herwald, and H. S. Kirschbaum, Electrical Analogy Methods Applied to Servomechanism Problems, *Trans. AIEE,* **65**: 91–96, 1946.

———, and R. H. MacNeal, Beam Vibration Analysis with the Electric-Analog Computer, *J. Applied Mechanics,* **16**, Paper 49SA3, 1949.

———, and C. H. Wilts, Application of Electric Analog Computers, *J. Applied Mechanics,* 1949.

———, C. H. Wilts, and B. N. Locanthi, Application of the California Institute of Technology Electric Analog Computer to Non-Linear Mechanics and Servomechanisms, *Trans. AIEE* (T 9165), **68**, 1949.

McCool, W. A., An Improved Analog Computing Circuit, *NRL Rept.* P-3423 (unclassified), Feb. 23, 1949.

McDonald, D., Analog Computers for Servo Problems, *Rev. Sci. Instruments,* **21**: 154, 1950.

Menely, C. A., and L. D. Morril, Linear Electronic Analog Computer Design, *NEC Proc.,* 1949.

Miller, S. E., Sensitive D-c Amplifier with A-c Operation, *Electronics*, November, 1941.

Minorsky, N., *Introduction to Non-Linear Mechanics*, Edwards Bros., Inc., Ann Arbor, Mich., 1947.

Murray, F. J., *Theory of Mathematical Machines*, King's Crown Press, New York, 1947.

Mynall, D. J., Electronic Analog Computing, *Elec. Eng.*, **19**: 178, 214, 254, 283, 1947.

———, Electronic Function Generator, *Nature,* **159** (No. 4048): 243, May 31, 1947.

Nettleton, L. A., and F. E. Dole, Potentiometers, *Rev. Sci. Instruments,* **17**: 356, 1946.

Paschkis, V., Comparison of Long-time and Short-time Analog Computers, *Proc. AIEE,* T 913. Reprint from *Trans. AIEE,* **68**, 1949.

Pauling, L., and E. B. Wilson, *Introduction to Quantum Mechanics*, McGraw-Hill, New York, 1935.

Philbrick, G. A., Designing Industrial Controllers by Analog, *Electronics*, June, 1948.

———, The High-speed Analog Computer as Applied in Industry, *Paper* 195 submitted by ASME Committee on Theory for presentation at the spring meeting, New London, Conn., May, 1949.

———, W. T. Stark, and W. C. Schaffer, Electronic Analog Studies for Turboprop Control Systems, *SAE Quart. Trans.*, **2** (No. 2), April, 1948.

Ragazzini, J. R., R. H. Randall, and F. A. Russell, Analysis of Problems in Dynamics by Electric Circuits, *Proc. IRE*, May, 1947, pp. 444–452.

Reeves Instrument Corporation, *REAC Manual*, New York.

Reich, H., *Principles of Electron Tubes*, McGraw-Hill, New York, 1941.

Rosenberg, P., Sinusoidal Potentiometers Types RL10CB, RL10CD, RL10E, RL14, *MIT RL Rept.* 423, Aug. 16, 1943.

Rosser, J. B., R. R. Newton, and G. L. Gross, *Mathematical Theory of Rocket Flight*, McGraw-Hill, New York, 1947.

Sack, H. S., R. T. Beyer, G. H. Miller, and J. W. Trischka, Special Magnetic Amplifiers and Their Use in Computing Circuits, *Proc. IRE*, **35** (No. 11): 1375–1382, November, 1947.

Seamans, R. C., B. G. Bromberg, and L. E. Payne, *Application of the Performance Operator to Aircraft Automatic Control, MIT DIC Rept.*, 1947 (Instrumentation Laboratory).

Shannon, C. E., Mathematical Theory of the Differential Analyzer, *J. Math. Phys.*, **20**: 337–354, 1941.

Shannon, W., Electronic Computers, *Electronics*, August, 1946.

Shaper, H. B., Electrodynamic Direct-inking Pen, *Electronics*, March, 1946.

Smith, F. L., Editor, *The Radiotron Designer's Handbook*, 3d ed., Radio Printing Press Pty, Ltd., Sydney, Australia. Reproduced and distributed in the United States by RCA Manufacturing Company, Inc., 1941.

Smythe, W. R., *Static and Dynamic Electricity*, McGraw-Hill, New York, 1939.

Sunstein, D. E., Photoelectric Waveform Generator, *Electronics*, **22**: 100, February, 1949.

Terman, F. E., *Radio Engineers' Handbook*, McGraw-Hill, New York, 1943.

Valley, G. E., and H. Wallman, *Vacuum Tube Amplifiers*, MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948.

Volkers, W. K., Ultra-high Gain Direct-coupled Amplifier Circuits, lecture presented at the IRE National Convention, New York, 1950.

Von Mises, R., *Theory of Flight*, McGraw-Hill, New York, 1945.

Wallman, H., An Electronic Integral-Transform Computer and the Practical Solution of Integral Equations, *J. Franklin Inst.*, **250**: 45, 1950.

Williams, F. C., and F. J. V. Ritson, Electronic Servo Simulators, *J. Inst. Elec. Engrs. (London)*, **94**, Part IIA, December, 1947. (Abstract Part I, p. 547, 1947.)

Winson, J., Solution of Aeroelastic Problems by Electronic Analog Computation, *J. Aeronaut. Sci.*, **17**: 385, 1950.

EXAMPLES OF DIFFERENTIAL ANALYZER COMPUTATIONS

The following papers deal with applications of mechanical differential analyzers but will be of considerable interest to users of electronic analog computers as well.[1]

[1] The authors are indebted to Professor V. C. Rideout of the University of Wisconsin for providing them with these references.

Brainerd, J. G., and H. W. Emmons, Effect of Variable Viscosity on Boundary Layers, with a Discussion of Drag Measurements, *J. Applied Mechanics*, **9** (No. 1): A1–6, March, 1942.

Brainerd, J. G., and C. N. Weygandt, Unsymmetrical Self-Excited Oscillations in Certain Simple Non-Linear Systems, *Proc. IRE*, **24** (No. 6): 414, June, 1936.

Bush, V., and S. H. Caldwell, Thomas-Fermi Equation Solution by the Differential Analyzer, *Phys. Rev.*, **38** (No. 10): 1898–1902, Nov. 15, 1931.

Copple, C., D. R. Hartree, A. Porter, and H. Tyson, The Evaluation of Transient Temperature Distributions in a Dielectric in an Alternating Field, *J. IEE*, **85** (No. 511): 56, July, 1939.

Gager, F. M., and J. B. Russell, Jr., A quantitative Study of the Dynatron, *Proc. IRE*, **23** (No. 12): 1536, December, 1935.

Hartree, D. R., Approximate Wave Functions and Atomic Field for Mercury, *Phys. Rev.* **46**: 738, Oct. 15, 1934.

———, On an Equation Occurring in Falkner and Skon's Approximate Treatment of the Equations of the Boundary Layer, *Proc. Cambridge Phil. Soc.*, **33**: 223, 1937.

——— and A. Porter, The Applications of the Differential Analyzer to Transients on a Distortionless Transmission Line, *J. Inst. Elec. Engrs.*, **83** (No. 503): 648, November, 1938.

——— and J. R. Womersley, A Method for the Numerical or Mechanical Solution of Certain Types of Partial Differential Equations, *Proc. Roy. Soc.* (*London*), **161** (No. 906): 355–366, August, 1937.

Lamaitree, G., and M. S. Vallarta, On the Allowed Cone of Cosmic Radiation, *Phys. Rev.*, **50** (No. 6): 493, Sept. 15, 1936.

Manning, M. F., and J. Millman, Self-consistent Field for Tungsten, *Phys. Rev.*, **49**: 848, June 1, 1936.

Meyers, D. M., D. R. Hartree, and A. Porter, The Effect of Space-Charge on the Secondary Current in a Triode, *Proc. Roy. Soc.* (*London*), A, **158**: 23, 1937.

Roder, H., Effect of Tuned Circuits upon a Frequency Modulated Signal, *Proc. IRE*, **25** (No. 12): 1617, December, 1937.

Sears, F. W., A Study of Electron Orbits in the Barkhausen-Kurz Effect, *J. Franklin Inst.*, **209** (No. 4): 459, April, 1930.

Summers, I. H., and J. B. McClure, Progress in the Study of System Stability, *Trans. AIEE*, **49**: 132–158, April, 1930.

Teplow, L., Stability of Synchronous Motors under Variable Load, *Gen. Elec. Rev.*, **31**: 356, June, 1928.

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